82. On the Zero-Free Region of Dirichlet's L-Functions

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1. Let $L(s,\chi)$ $(s=\sigma+it)$ be the Dirichlet L-function for a Dirichlet character χ. We denote by $\mathcal{Z}(T)$ the set of all zeros in the region $0<\sigma<1$, $|t|\leqq T$ of all primitive L-functions of modulus $\leqq T$. Then the fundamental result on the zero-free region for $L(s,\chi)$ is

Theorem. *For any $\rho\in\mathcal{Z}(T)$ we have*

$$\mathrm{Re}\,\rho\leqq 1-c_0(\log T)^{-1}, \tag{1}$$

save for at most one zero, where c_0 is an effectively computable positive constant. This (possibly existing) exceptional zero β_1 is real and simple, and comes from $L(s,\chi_1)$ with a unique real character χ_1. Further there exists a function $c(\varepsilon)>0$ such that for any $\varepsilon>0$

$$\beta_1\leqq 1-c(\varepsilon)T^{-\varepsilon}. \tag{2}$$

(1) is the Page-Landau theorem, and (2) is Siegel's theorem in which $c(\varepsilon)$ is not effectively computable. The purpose of the present note is to modify the argument of our preceding note [1] so as to prove this theorem without appealing to the deep function-theoretical properties of $L(s,\chi)$. The details will appear elsewhere.

2. In what follows we assume always that T is sufficiently large.

Lemma 1. *Uniformly for $0\leqq\sigma\leqq 1$ and for $\chi(\mathrm{mod}\ q)$ we have*

$$L(s,\chi)\ll(q(|t|+1))^{1-\sigma}\log(q(|t|+2)).$$

If χ is principal, the region $|s-1|\leqq 1/2$ should be excluded.

Lemma 2. *For any $\rho\in\mathcal{Z}(T)$ we have*

$$\mathrm{Re}\,\rho\leqq 1-T^{-3}.$$

Lemma 1 is not the best among results of this type, but the above assertion can be proved only by the partial summation. Lemma 2 is quite rough, but this is important in our procedure. To prove it let $L(\rho,\chi)=0$. Either if χ is complex or if $|\mathrm{Im}(\rho)|\geqq T^{-2}$, then the argument of [2, pp. 43–44] does work also for $L(s,\chi)$. So in these cases we have $\mathrm{Re}\,\rho\leqq 1-T^{-3}$. Otherwise let $a(n)=\sum_{d|n}\chi(d)$. Then $a(n)\geqq 0$ and $a(n^2)\geqq 1$. So by Lemma 1, we have

$$N^{1/2}\ll\sum_{n\leqq N}a(n)(\log N/n)^2$$
$$=2L(1,\chi)N+O(T(\log T)^2).$$

Hence $L(1,\chi)\gg T^{-1}(\log T)^{-2}$, from which the desired assertion follows easily.

Now let $1-\delta+i\tau \in \mathscr{Z}(T)$ be a zero of $L(s, \psi)$. We may assume $T^{-3} \leqq \delta \leqq 1/4$ by the obvious reason. Next let $f(n)$ be the coefficient of the Dirichlet series for $\zeta(s)L(s+\delta+i\tau, \psi)$. And let us apply the Selberg sieve to the sequence $\{|f(n)^2|\}$. The situation is quite similar to the corresponding part of [1], so hereafter we adopt the notations of [1].

Lemma 3. *Let θ_d be defined by* (1) *of* [1], *and let us put*

$$V(N) = \sum_{n \leqq N} |f(n)|^2 \Big(\sum_{d|n} \theta_d\Big)^2.$$

Then we have, provided $R \geqq T^B$ and $N \geqq R^4$,

$$V(N) \ll_B \delta N.$$

To prove this we observe that for $\sigma > 1$

$$\sum_{n=1}^{\infty} |f(n)|^2 n^{-s}$$
$$= \zeta(s)L(s+2\delta, \psi_0)L(s+\delta+i\tau, \psi)L(s+\delta-i\tau, \bar{\psi})L(2(s+\delta), \psi_0)^{-1},$$

where $\psi_0 = \psi\bar{\psi}$. Lemma 1 and this yield, on the conditions given above,

$$V(N) \ll_B L(1+2\delta, \psi_0)\,|L(1+\delta+i\tau, \psi)|^2\, G(R)^{-1} N.$$

On the other hand the same combination gives, provided $R \geqq T^B$,

$$G(R) \gg_B \delta^{-1} L(1+2\delta, \psi_0)\,|L(1+\delta+i\tau, \psi)|^2.$$

Thus the assertion of the lemma follows.

3. Now we give a brief proof of the theorem. Let ω_d be defined by

$$\sum_{d|n} \omega_d = \Big(\sum_{d|n} \theta_d\Big)\Big(\sum_{d|n} \lambda_d\Big),$$

where θ_d and λ_d are defined by (1) and Lemma 4 of [2], respectively. And we put

$$M(s, \chi) = \sum_{d \leqq z^2 R} \omega_d \chi(d) d^{-s} \prod_{p|d} \Big(1 + \frac{\psi(p)}{p^{\delta+i\tau}} - \frac{\chi\psi(p)}{p^{s+\delta+i\tau}}\Big).$$

Then we have, for $\sigma > 1$,

$$L(s, \chi)L(s+\delta+i\tau, \chi\psi)M(s, \chi) = 1 + \sum_{z \leqq n} \chi(n) f(n) \Big(\sum_{d|n} \omega_d\Big) n^{-s}.$$

Hereafter let χ be non-principal, and let $K(s, \chi) = L(s, \chi)L(s+\delta+i\tau, \chi\psi)$. Then $K(s, \chi)$ is regular for $\sigma > 0$. Now let $\rho = \beta + i\gamma$ be a zero of $K(s, \chi)$ such that $|\gamma| \leqq T$ and $1-\beta \leqq 1/4$. And we set $z = T^{4A}$, $R = T^A$, $X = T^{10A}$ with a large constant A. Then we get, by Lemma 1 and by a routine reasoning,

$$1 \ll \sum_{z \leqq n} |f(n)| \Big|\sum_{d|n} \omega_d\Big| n^{-\beta} e^{-n/X}.$$

Hence by Lemma 4 of [1] and by Lemma 2 above we get, just as in [1],

$$(3) \qquad 1 \ll \delta T^{20A(1-\beta)} \log T.$$

Now either if ψ is complex, or if ψ is real and $\tau \neq 0$, or if ψ is real, $\tau = 0$ and $1-\delta$ is a multiple zero, then obviously $K(1-\delta+i\tau, \psi) = 0$. Namely in these cases we can set $\rho = 1-\delta+i\tau$ in (3), and we find $\delta \gg (\log T)^{-1}$. In the remaining case we may assume that ψ is real, and $1-\delta$ is a simple zero of $L(s, \psi)$. Then we may further assume that $\delta \leqq c'(\log T)^{-1}$ with a certain sufficiently small constant $c'>0$. Then (3) implies all other zeros of $\mathcal{Z}(T)$ are in the range $\sigma \geqq 1 - c''(\log T)^{-1}$ with an effective $c''>0$. Now only the exceptional zero remains. So in (3) set $\beta_1 = 1-\delta$. Then (3) is the Deuring-Heilbronn phenomenon which, as is well-known, implies Siegel's theorem. This ends the proof of the theorem.

Added in Proof (Dec. 13, 1978). In the mean time the author has realized that the whole argument of [1] and the present note (including Lemma 4 of [1]) can be made independent of the theory of functions. Thus we have now proved *elementary* the most fundamental results on the zeta- and L-functions. For the details see our forthcoming paper entitled "*An elementary proof of Vinogradov's zero-free region for the Riemann zeta-function*".

References

[1] Y. Motohashi: On Vinogradov's zero-free region for the Riemann zeta-function. Proc. Japan Acad., **54**A, 300–302 (1978).
[2] E. C. Titchmarsh: The Theory of the Riemann Zeta-Function. Oxford Univ. Press (1951).