71. *Sylow Subgroups in a Pair of Locally Finite Groups*

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(Communicated by Kôsaku YOSIDA, M. J. A., Nov., 13, 1978)

Introduction. Following Z. Goseki [2] we define a collection (A, B, f, g) as follows: Let A and B be groups. If there are homomorphisms f and g such that $\xrightarrow{g} A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{f}$ is exact, we say that the collection (A, B, f, g) is well defined. Suppose (C, D, f_1, g_1) is well defined where C and D are subgroups of A and B respectively. If $f_1 = f$ on C and $g_1 = g$ on D then we call (C, D, f_1, g_1) a subgroup of (A, B, f, g) and denote it by (C, D, f, g). If $C \lhd A$ and $D \lhd B$ then (C, D, f, g) is a normal subgroup of (A, B, f, g). Goseki [2] states that if (C, D, f, g) is a subgroup such that C is a Sylow p subgroup of A then D is a Sylow p subgroup of B.

We prove that this statement does not hold in general but does hold for a wide class Γ_p of groups which contains for example periodic soluble linear groups and FC groups (locally normal groups). π will always denote a set of primes and π' its complementary set.

The following fact is a direct consequence of Zorn's lemma. "Let G be any group. Then every π subgroup of G is contained in a maximal π subgroup of G". In particular G possesses a maximal π subgroup, we shall refer to such as S_π subgroups.

Definitions. 1) *A local system for a group G is a set Σ of subgroups such that every finite subset of G is contained in some member of Σ.*

2) *A group is locally finite if it has a local system consisting of finite subgroups.*

In this paper all groups will be locally finite and all local systems will consist of finite subgroups.

Following [8] we define an S_π subgroup to be good if it reduces into a local system.

Definition. *An S_π subgroup P of G is good with respect to a local system Σ if for each $X \in \Sigma$ we have that $P \cap X$ is an S_π subgroup of X. We say that P is good if there is some local system with respect to which it is good.*

It is not hard to prove ([8], Proposition 1.12) that

Proposition 1. *If N is a normal subgroup of G, P is an S_π subgroup of G which is good with respect to Σ and $P \cap X$ is a Hall π subgroup of X for each X then $P \cap N$ and PN/N are good S_π subgroups* *of N and G/N respectively.*

Note. A Hall π subgroup of a finite group is a π subgroup whose index is a π' number.

In this paper we show that Goseki's remark mentioned above is true for infinite groups provided that only "good" subgroups are used, and give a counter example to show that it need not hold in general (in fact it seems likely that it is true only for good S_π subgroups). We then briefly discuss the class Γ_p of groups all of whose S_p subgroups are good, showing that it contains many important classes of groups. Let

$$\Gamma = \bigcap_{\substack{\text{all primes} \\ p}} \Gamma_p.$$

Remark. We remark that most of the results of [2] and [12] can easily be extended to infinite Γ groups. For example Theorems 2 and 3 of [2] and Theorems 1 and 2 of [2(i)].

§ 1. The following example shows that if (A_p, T_p, f, g) is a subgroup of (A, B, f, g) where A_p is an S_p subgroup of A then T_p need not be an S_p subgroup of B.

Example. Let A be the standard restricted wreath product.

$$A = C_q \wr (C_{p^\infty} \times C_p)$$

where C_p and C_q are cyclic groups of orders p and q respectively and C_{p^∞} is a Prüfer group.

Let $B = E_{q^\infty} \times (C_{p^\infty} \times C_p)$ where E_{q^∞} is an infinite elementary abelian q group. Then A has a normal subgroup $M \cong E_{q^\infty}$ which is the base group of the wreath product and B has a normal subgroup $N \cong C_{p^\infty} \times C_p$. Take f to be the natural homomorphism with kernel M and image N. Similarly let g have kernel N and image M.

Let A_p be an S_p subgroup of A which is isomorphic to C_{p^∞} (by Hartley [3], Theorem A).

Let $T_p = f(A_p)$ then $f(A_p) = N \cap T_p$ and $g(T_p) = 1 = M \cap A_p$. Thus (by [2], Lemma 1) (A_p, T_p, f, g) is well defined. But T_p clearly is not an S_p subgroup of B.

This shows that Goseki's lemma ([2], Lemma 5) which holds for finite groups does not hold in general for infinite groups.

We now prove that Goseki's theorem holds for all good S_p subgroups.

Theorem. *Let (A_p, T_p, f, g) be a subgroup of (A, B, f, g). Then if A_p is a good S_p subgroup of A, it follows that T_p is an S_p subgroup of B.*

Before the proof we need a lemma.

Lemma 1. *Let S be an S_p subgroup of G such that for some normal subgroup N we have that $S \cap N$ and SN/N are S_p subgroups of N and G/N respectively, then S is an S_p subgroup of G.*

Proof. If $S \cap N$ is an S_p subgroup of N and SN/N an S_p subgroup of G/N and S is not an S_p subgroup of G then let $R \supseteq S$ be an S_p subgroup of G. Then $R \cap N$ and RN/N are p subgroups of N and G/N respectively. Thus

$$RN = SN \quad \text{and} \quad R \cap N = S \cap N$$

so

$$R = S.$$

The proof of the theorem. By the above lemma T_p is an S_p subgroup of B if T_pN/N and $T_p \cap N$ are S_p subgroups of $B/N \cong M$ and N respectively.

Now since (A_p, T_p, f, g) is well defined, by Goseki [2] we have

$$g(T_pN) = A_p \cap M$$

and

$$f(A_pM) = T_p \cap N.$$

By Proposition 1 above we have $A_p \cap M$ and A_pM/M are S_p subgroups of M and A/M respectively. Since f induces an isomorphism between A/M and N and g between B/N and M, we can deduce that T_pN/N and $T_p \cap N$ are S_p subgroups of B/N and N respectively.

By the lemma it follows that T_p is an S_p subgroup of B as asserted.

Note. We have not been able to determine whether T_p is necessarily good. If A and B are soluble however (or even π-ascendant, see [8]). Theorem A1 of [8] may be used to show that T_p must be good.

§2. **Definition.** *The class Γ_p consists of those groups all of whose S_p subgroups are good. Let*

$$\Gamma = \bigcap_{\text{all primes}} \Gamma_p.$$

B. Hartley [5] defines the class U as follows.

Definition. *$G \in U$ if and only if the following conditions are satisfied.*

(U1) *The locally finite group G has a finite series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ such that G_i/G_{i+1} is locally nilpotent for $i = 1, \cdots, n-1$.*

(U2) *If H is any subgroup of G and π any set of primes then all S_π subgroups of H are conjugate.*

Theorem. *U is contained in Γ.*

Proof. Let $G \in U, P$ be any S_π subgroup and $N \lhd G$. Then by Hartly ([5], Lemma 2.14) $P \cap N$ and PN/N are S_π subgroups of N and G/N respectively. Now by A. Rae ([8], Theorem A1) it follows that since P covers every π factor of G, it is good. Thus $G \in \Gamma$.

Definition. *A group $G \in \mathcal{L}$ if G has a subnormal local system Σ (i.e., one such that if $X \subseteq Y$ where X, Y are members of Σ then X is a subnormal subgroup of Y).*

In [9] it is shown that the class $\mathcal{L}$ contains the class of periodic locally normal groups [6] and is in Σ.

In addition Wehrfritz [11] shows that the class of periodic locally soluble linear groups is in the class U. Stonehewer [10] shows that locally soluble groups having a locally nilpotent subgroups of finite index and McDougall [7] that metabelian groups satisfying the minimal condition for normal subgroups are in the class U. Thus all these classes are contained in Γ.

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