64. A Generalization of Local Class Field Theory by Using K-Groups. II

By Kazuya KATO
Department of Mathematics, Faculty of Science,
University of Tokyo, Hongo, Tokyo

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In § 1, we study abelian extensions of complete discrete valuation fields whose residue fields are function fields in one variable over finite fields. In § 2, we give a generalization of the result of Part I (K. Kato [5]). The detail will appear elsewhere. Some similar results were obtained independently by A. N. Parsin [8].

§ 1. Let F be an algebraic function field in one variable over a finite field, and K a complete discrete valuation field with residue field F. We shall define the "K_2-idele group" of K, which is a K_2-version of the idele group in the usual class field theory. For this purpose, let $\mathfrak{P}(F)$ be the set of all normalized discrete valuations of F. For each $v \in \mathfrak{P}(F)$, let F_v be the completion of F with respect to v, and K_v the extension of K which is complete with respect to a discrete valuation and characterized by the following properties; the restriction of the normalized valuation ord_{K_v} of K_v to K coincides with the normalized valuation of K, and the residue field of K_v is isomorphic to F_v over F. Such K_v exists and is essentially unique by Grothendieck [4, Chap. 0 § 19]. The K_2-idele group will be defined as a kind of restricted direct product of the groups $K_2(K_v)$ $(v \in \mathfrak{P}(F))$. To define this, take a triple (A, π, S) consisting of a subring A of the valuation ring O_K of K, a prime element π of K, and a non-empty finite subset S of $\mathfrak{P}(F)$, such that (1) $\pi \in A$ and (2) the canonical homomorphism $A/\pi A \to F$ is injective and its image is $\bigcap_{v \in \mathfrak{P}(F)-S} O_v$. Here, O_v denotes the valuation ring of v for each v. Such a triple (A, π, S) exists. For each $v \in \mathfrak{P}(F) - S$, let m_v be the maximal ideal of A induced by v, and let $A_v = \varprojlim A/m_v^n$. Then, A_v is canonically embedded in K_v. For each $v \in \mathfrak{P}(F)$ and for each $n \geqq 1$, let $K_2(K_v)^{(n)}$ be the subgroup of $K_2(K_v)$ generated by all elements of the form $\{1+x, y\}$ such that $x \in K_v$, $\mathrm{ord}_{K_v}(x) \geqq n$ and $y \in K_v^*$. For $v \in \mathfrak{P}(F) - S$, let I_v be the subgroup of $K_2(K_v)$ generated by all elements of the form $\{x, y\}$ such that $x, y \in A_v[\pi^{-1}]^*$. (The notation $*$ denotes the group of all invertible elements of a ring.) Now, we call an element $(a_v)_{v \in \mathfrak{P}(F)}$ of $\prod_{v \in \mathfrak{P}(F)} K_2(K_v)$ a K_2-idele of K if and only if for each $n \geqq 1$, there is a finite subset S_n of $\mathfrak{P}(F)$ containing S such that $a_v \in I_v \cdot K_2(K_v)^{(n)}$ for any $v \in \mathfrak{P}(F) - S_n$. We denote by $\varLambda_K$ the group of all K_2-ideles of K. Whether an element of $\prod_{v\in\mathfrak{P}(F)} K_2(K_v)$ is a K_2-idele of K or not is independent of the choice of the triple (A, π, S), and the image of the canonical homomorphism $K_2(K)\to\prod_{v\in\mathfrak{P}(F)} K_2(K_v)$ is contained in $\varLambda_K$. We denote Coker $(K_2(K)\to\varLambda_K)$ by C_K, which is an analogue of the idele class group in the usual class field theory. We endow $\varLambda_K$ and C_K with the following topologies. For each $n\geqq 1$, let $\varLambda_K^{(n)}=\varLambda_K\cap\prod_{v\in\mathfrak{P}(F)} K_2(K_v)^{(n)}$. We endow $\varLambda_K/\varLambda_K^{(n)}$ with the strongest topology which is compatible with the group structure and for which the mapping $\prod_{v\in S} K_2(K_v)\times\prod_{v\notin S} I_v\to\varLambda_K/\varLambda_K^{(n)}$ is continuous. Here the topologies of $K_2(K_v)$ are the ones defined in [5, § 4], and those of I_v are the ones induced by the topologies of $K_2(K_v)$. This topology of $\varLambda_K/\varLambda_K^{(n)}$ is independent of the choice of the triple (A, π, S). We endow $\varLambda_K$ with the weakest topology for which the mappings $\varLambda_K\to\varLambda_K/\varLambda_K^{(n)}$ are continuous for all $n\geqq 1$. We endow C_K with the quotient of the topology of $\varLambda_K$. Now, we can state our result. For each $v\in\mathfrak{P}(F)$, let $\Phi_{K_v}: K_2(K_v)\to\mathrm{Gal}(K_v^{\mathrm{ab}}/K_v)$ be the canonical homomorphism of [5, Theorem 1].

Theorem 1. *Let F and K be as at the beginning of* § 1. *Then*:

(1) *There exists a unique continuous homomorphism*

$$\Phi: C_K\to\mathrm{Gal}(K^{\mathrm{ab}}/K)$$

for which the following diagram is commutative for every $v\in\mathfrak{P}(F)$.

$$\begin{array}{ccc} K_2(K_v) & \xrightarrow{\Phi_{K_v}} & \mathrm{Gal}(K_v^{\mathrm{ab}}/K_v) \\ \downarrow & & \downarrow \\ C_K & \xrightarrow{\Phi} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \end{array}$$

(2) *For each finite abelian extension L of K, Φ induces an isomorphism $C_K/N_{L/K}C_L\cong\mathrm{Gal}(L/K)$.*

(3) *The mapping $L\mapsto N_{L/K}C_L$ is a bijection from the set of all finite abelian extensions of K in a fixed algebraic closure of K to the set of all open subgroups of C_K of finite indices.*

§ 2.1. Here, we generalize the result of [5].

For any field k, let $\mathfrak{K}_n(k)$ $(n\geqq 0)$ be Milnor's K-groups defined in Milnor [7] (which were denoted by K_nk in [7]), i.e.,

$$\mathfrak{K}_n(k)=(\overbrace{k^*\otimes\cdots\otimes k^*}^{n\text{-times}})/J,$$

where J denotes the subgroup of the tensor product generated by all elements of the form $x_1\otimes\cdots\otimes x_n$ satisfying $x_i+x_j=1$ with i and j such that $i\neq j$. For any $x_1,\cdots,x_n\in k^*$, the element $x_1\otimes\cdots\otimes x_n \bmod J$ of $\mathfrak{K}_n(k)$ is denoted by $\{x_1,\cdots,x_n\}$. On the other hand, for any ring R, let $K_n(R)$ $(n\geqq 0)$ be Quillen's K-groups in Quillen [9]. If k is a field, there is a canonical homomorphism $\iota_k:\mathfrak{K}_n(k)\to K_n(k)$ (the product defined in Loday [6]). This ι_k is bijective when $n\leqq 2$, but not always so in the general case.

If E is a finite extension of a field k, there is a transfer map $K_*(E)$

$\to K_*(k)$ ([9, § 4]), which we denote by $N_{E/k}$. Concerning the $\mathfrak{K}$-groups, if E is

(*) *a composite field of a finite abelian extension and a finite purely in separable extension*

of k, we can define a canonical homomorphism $\mathfrak{N}_{E/k} : \mathfrak{K}_*(E) \to \mathfrak{K}_*(k)$ characterized by the following:

(1) If $k \subset F \subset E$ and the extension E/k is of the type (*) above, $\mathfrak{N}_{F/k} \circ \mathfrak{N}_{E/F} = \mathfrak{N}_{E/k}$.

(2) If E is a normal extension of k of a prime degree, $\mathfrak{N}_{E/k}$ coincides with the homomorphism $N_{\alpha/k}$ of Bass and Tate [1, § 5] for any choice of α such that $E = k(\alpha)$. These homomorphisms $N_{E/k}$ and $\mathfrak{N}_{E/k}$ satisfy $N_{E/k} \circ \iota_E = \iota_k \circ \mathfrak{N}_{E/k}$.

Now, our results are the following two theorems.

Theorem 2. *Let $n \geqq 0$ and let $F_0, \cdots, F_n$ be fields having the following properties:*

(1) *F_0 is a finite field.*

(2) *For each $i = 1, \cdots, n$, F_i is complete with respect to a discrete valuation and the residue field of F_i is F_{i-1}.*

Then, there exists a unique system $(\Phi_K)_K$ which assigns to each finite extension K of F_n a homomorphism $\Phi_K : \mathfrak{K}_n(K) \to \mathrm{Gal}(K^{ab}/K)$ satisfying the following conditions (3) *and* (4).

(3) *Let K and L be finite extensions of F_n, and f an F_n-homomorphism $K \to L$. If the extension f is of the type* (*) (*resp. is separable*), *the following diagram* (i) (*resp.* (ii)) *is commutative. Here, the vertical arrows in the diagrams are the ones induced by the extension f.*

$$\text{(i)}\quad \begin{array}{ccc} \mathfrak{K}_n(L) & \xrightarrow{\Phi_L} & \mathrm{Gal}(L^{ab}/L) \\ {\scriptstyle \mathfrak{N}_{L/K}}\downarrow & & \downarrow \text{restriction,} \\ \mathfrak{K}_n(K) & \xrightarrow{\Phi_K} & \mathrm{Gal}(K^{ab}/K) \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} \mathfrak{K}_n(K) & \xrightarrow{\Phi_K} & \mathrm{Gal}(K^{ab}/K) \\ \downarrow & & \downarrow \text{transfer.} \\ \mathfrak{K}_n(L) & \xrightarrow{\Phi_L} & \mathrm{Gal}(L^{ab}/L) \end{array}$$

(4) *Let π_i be a lifting to F_n of a prime element of F_i for each $i = 1, \cdots, n$. Then, the image of $\Phi_{F_n}(\{\pi_1, \cdots, \pi_n\})$ under the canonical homomorphism $\mathrm{Gal}(F_n^{ab}/F_n) \to \mathrm{Gal}(F_0^{ab}/F_0)$ coincides with the Frobenius automorphism over F_0.*

Furthermore, this system $(\Phi_K)_K$ satisfies:

(5) *If the extension f in* (3) *is abelian, the diagram* (i) *induces an isomorphism $\mathfrak{K}_n(K)/\mathfrak{N}_{L/K}\mathfrak{K}_n(L) \cong \mathrm{Gal}(L/K)$.*

Theorem 3. *Besides the hypothesis of Theorem* 2, *suppose that the characteristic $\mathrm{ch}(F_n)$ of F_n is $p > 0$. Then, there exists a system $(\Upsilon_K)_K$ which assigns to each finite extension K of F_n a homomorphism $\Upsilon_K : K_n(K) \to \mathrm{Gal}(K^{ab}/K)$ satisfying the following conditions.*

(1) *Let K and L be finite extensions of F_n and f an F_n-homomorphism $K \to L$. Then, the diagram* (i) (*resp. if f is separable, the diagram* (ii)) *in Theorem* 1 *is commutative when we replace $\mathfrak{K}_n$, $\mathfrak{N}_{L/K}$ and*

Φ by K_n, $N_{L/K}$ and Υ, respectively.

(2) *The composite $\Upsilon_K \circ \iota_K$ coincides with Φ_K in Theorem* 1.

§ 2.2. The construction of the homomorphism Φ_K in the case $\mathrm{ch}(F_n)=0$. The key tool is the following definition of a homomorphism called the *cohomological residue*. For any discrete valuation field k, let O_k be the valuation ring of k, m_k the maximal ideal of O_k, and $\bar{k}$ the residue field of k. Now, suppose that k and K are complete discrete valuation fields of characteristic zero such that $k \subset K$ and such that the following conditions (a), (b) and (c) are satisfied.

(a) The inclusion $k \subset K$ satisfies $O_k \subset O_K$ and $m_k \subset m_K$.

(b) $\bar{K}$ is a henselian discrete valuation field whose valuation ring contains $\bar{k}$ and whose residue field $\bar{\bar{K}}$ is a finite extension of $\bar{k}$.

(c) The transcendental degree of $\bar{K}$ over $\bar{k}$ is one.

(The conditions (b) and (c) are satisfied, for example, if $\bar{K}$ is the algebraic closure of $\bar{k}(X)$ in the field of formal power series $\bar{k}((X))$.) Fix integers $i \geqq 0$, $m \geqq 1$, and r. We now define a homomorphism $H^{i+1}(K, \mu_m^{\otimes(r+1)}) \rightarrow H^i(k, \mu_m^{\otimes r})$, called the cohomological residue. Here $\mu_m^{\otimes r}$ denotes the r-th tensor power of μ_m over $\boldsymbol{Z}/m\boldsymbol{Z}$.

First, let $t_{K/k}: K^* \rightarrow \boldsymbol{Z}$ be the homomorphism characterized by the following properties:

(1) If $x \in O_K^*$, and if e denotes $\mathrm{ord}_K(\pi)$ for prime elements π of k, $t_{K/k}(x)=[\bar{\bar{K}}:\bar{k}]\cdot e \cdot \mathrm{ord}_{\bar{K}}(x \bmod m_K)$. (ord denotes the normalized additive valuation.)

(2) $t_{K/k}(k^*)=0$.

Next, let k_s be the algebraic closure of k. We can show that there is a $\mathrm{Gal}(k_s/k)$-homomorphism $T_{K/k}: (k_s \otimes_k K)^* \rightarrow \boldsymbol{Z}$ characterized by the following property: If E is a finite extension of k, and if $E \otimes_k K = \prod_j K_j$ a finite product of fields, the restriction of $T_{K/k}$ to K_j^* coincides with $t_{K_j/E}$. On the other hand, we can deduce from the condition (9) that the composite field $k_s \cdot K$ over k is of cohomological dimension one (cf. Serre [10, Chap. II § 4]). By this, we obtain the desired cohomological residue as follows.

$$H^{i+1}(K, \mu_m^{\otimes(r+1)}) \cong H^i(k, \mu_m^{\otimes r} \otimes (k_s \otimes_k K)^*) \xrightarrow{\text{by } T_{K/k}} H^i(k, \mu_m^{\otimes r}).$$

Now, let $n \geqq 1$ and let $F_0, \cdots, F_n$ be as in Theorem 2. Suppose that $\mathrm{ch}(F_n)=0$. To construct the homomorphism Φ_K, we may assume that $K=F_n$ without loss of generality. Let X_K be the character group of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. Let $m \geqq 1$, and α the composite

$$(X_K)_m \otimes \mathfrak{K}_n(K)/\mathfrak{K}_n(K)^m \xrightarrow{c \otimes h_m^{(n)}} H^1(K, \boldsymbol{Z}/m\boldsymbol{Z}) \otimes H^n(K, \mu_m^{\otimes n}) \xrightarrow{\text{cup product}} H^{n+1}(K, \mu_m^{\otimes n}),$$

where $(X_K)_m$ denotes the kernel of the multiplication by m on X_K, c denotes the canonical isomorphism $(X_K)_m \cong H^1(K, \boldsymbol{Z}/m\boldsymbol{Z})$, and $h_m^{(n)}$

denotes the homomorphism $\mathfrak{K}_n(K)/\mathfrak{K}_n(K)^m \to H^n(K, \mu_m^{\otimes n})$ defined in the same way as Tate's Galois symbol (Tate [11]). On the other hand, let β be the homomorphism

$$\frac{1}{m}Z/Z \cong (X_{F_0})_m \to H^{n+1}(K, \mu_m^{\otimes n})\ ;\ \chi \mapsto c(\tilde{\chi}) \cup h_m^{(n)}(\{\pi_1, \cdots, \pi_n\}),$$

where $\tilde{\chi}$ denotes the canonical lifting of $\chi \in (X_{F_0})_m$ to $(X_K)_m$, π_i denotes a lifting of a prime element of F_i to K for each i, and $\cup$ denotes the cup product. This homomorphism β is independent of the choices of of such $\pi_1, \cdots, \pi_n$. By some computation of the homomorphisms $\mathfrak{N}_{L/K} : \mathfrak{K}_n(L) \to \mathfrak{K}_n(K)$ for finite cyclic extensions L/K, we can prove that the image of α is contained in the image of β. Furthermore, we can deduce from Lemma 1 below that β is injective. Hence, we have a canonical homomorphism

$$\gamma : (X_K)_m \otimes \mathfrak{K}_n(K)/\mathfrak{K}_n(K)^m \to \frac{1}{m}Z/Z.$$

When m varies, this γ induces the desired canonical homomorphism

$$\Phi_K : \mathfrak{K}_n(K) \to \mathrm{Gal}\,(K^{\mathrm{ab}}/K).$$

Lemma 1. *Let k and K be complete discrete valuation fields of characteristic zero such that $k \subset K$, and such that the above condition* (a) *and the following conditions* (d) *and* (e) *are satisfied.*

(d) *A prime element of k is still a prime element in K.*

(e) *There is an isomorphism $\theta : \bar{k}((X)) \cong \bar{K}$ over $\bar{k}$.*

Let τ be a lifting of $\theta(X)$ to O_K. Then, for any $i \geqq 0$, $m \geqq 1$, and r, the homomorphism

$$H^i(k, \mu_m^{\otimes r}) \to H^{i+1}(K, \mu_m^{\otimes (r+1)})\ ;\ x \mapsto x \cup h_m^{(1)}(\tau)$$

is injective.

Indeed, if $\bar{k}((X))$ is replaced by the algebraic closure $\bar{k}((X))^\circ$ of $\bar{k}(X)$ in $\bar{k}((X))$, the cohomological residue gives the left inverse of the above homomorphism. We can proceed from $\bar{k}((X))^\circ$ to $\bar{k}((X))$, essentially because any finitely generated subring A of $\bar{k}((X))$ over $\bar{k}((X))^\circ$ has a ring homomorphism $A \to \bar{k}((X))^\circ$ over $\bar{k}((X))^\circ$.

§2.3. The construction of the pro-p-part of the homomorphism Υ_K int he case ch $(F_n)=p>0$. The main tool is the following generalization of the definition of the residue homomorphism by using K-groups. Let $A \to B$ be a flat homomorphism between commutative rings. Suppose that π is a non-zero-divisor in B such that $B/\pi B$ is finitely generated and projective as an A-module. Let H be the category of B-modules which admit a resolution of length 1 by finitely generated projective B-modules and on which the action of π is nilpotent. Then, we can define a homomorphism $K_{q+1}(B[\pi^{-1}]) \to K_q(A)$ for each $q \geqq 0$, as the composite of the homomorphism $K_{q+1}(B[\pi^{-1}]) \to K_q(H)$ defined by the localization theorem for projective modules of Grayson [3], and the homomorphism $K_q(H) \to K_q(A)$ defined by regarding the objects of H

as finitely generated and projective A-modules. By replacing A and B by $A[T]/(T^m)$ and $B[T]/(T^m)$ respectively for each $m\geqq 1$, we obtain a homomorphism, called the residue,

$$\mathrm{Res}_{q+1}: \hat{C}K_{q+1}(B[\pi^{-1}])\to\hat{C}K_q(A),$$

where $\hat{C}K_q(R)$ denotes

$$\varprojlim \mathrm{Ker}\,(K_q(R[T]/(T^m))\to K_q(R))$$

for any ring R as in Bloch [2].

Now, let F_i $(0\leqq i\leqq n)$ be as in Theorem 3, and let $K=F_n$. We give here the pro-p-part of the homomorphism Υ_K. For each i, fix a ring homomorphism $\theta_i: F_i\to O_{F_{i+1}}$ such that $\theta_i(x) \bmod m_{F_{i+1}}=x$ for all $x\in F_i$. We apply the above definition of the residue to the case in which $A=F_i$, $B=O_{F_{i+1}}$, and π is a prime element of F_{i+1}. Then, the composite Θ;

$$\hat{C}K_{n+1}(F_n)\xrightarrow{\mathrm{Res}_n}\hat{C}K_n(F_{n-1})\xrightarrow{\mathrm{Res}_{n-1}}\cdots\longrightarrow\hat{C}K_1(F_0)\xrightarrow{\text{transfer}}\hat{C}K_1(\boldsymbol{Z}/p\boldsymbol{Z})$$

is independent of the choices of θ_i $(0\leqq i<n)$. On the other hand, for any commutative ring R of characteristic p, let $W^{(p)}(R)$ be the group of p-Witt vectors regarded as a subgroup of $\hat{C}K_1(R)$ ([2, I § 1 (3.2)]). Then Θ induces a pairing

$$W^{(p)}(K)\otimes K_n(K)\to W^{(p)}(\boldsymbol{Z}/p\boldsymbol{Z})\subset\hat{C}K_1(\boldsymbol{Z}/p\boldsymbol{Z}); \qquad w\otimes a\mapsto\Theta(\{w, a\}),$$

and for each r if $\mathcal{F}$ denotes the Frobenius homomorphism, a pairing

$$\Theta_r: W_r(K)/(1-\mathcal{F})W_r(K)\otimes K_n(K)\to W_r(\boldsymbol{Z}/p\boldsymbol{Z})=\boldsymbol{Z}/p^r\boldsymbol{Z}.$$

When r varies, by Witt theory [12], these homomorphism Θ_r give a homomorphism from $K_n(K)$ to the pro-p-part of $\mathrm{Gal}\,(K^{\mathrm{ab}}/K)$, which is the pro-$p$-part of the homomorphism Υ_K.

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