62. A Complex Analogue of the Generalized Minkowski Problem. II

By Mikio ISE

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1. In this note we continue our studies achieved in [1] for some type of non-linear partial differential equations of the determinant type over the complex n-projective space P_C^n. In that previous note we were concerned the *real hessian* of the unknown real-valued function φ on P_C^n, while, here, we shall deal with those concerning the *complex hessian* of the real-valued function φ on P_C^n. Further, we will mention the relation between the result of the note [1] and that of the present one.

2. Let us now denote by (M, g) a compact smooth hermitian manifold with the hermitian metric g, and fix our notation adopted in what follows: $\mathcal{F}_R(M)$ designates the space of all real-valued smooth functions on M, $\mathcal{X}(M)$ that of all complex-valued smooth vector fields on M; whereby $\mathcal{X}(M)$ is decomposed directly into the two spaces $\mathcal{X}^{1,0}(M)$ and $\mathcal{X}^{0,1}(M)$: $\mathcal{X}(M)=\mathcal{X}^{1,0}(M)+\mathcal{X}^{0,1}(M)$, where $\mathcal{X}^{1,0}(M)$ (resp. $\mathcal{X}^{0,1}(M)$) denotes the space of vector fields of type $(1,0)$ (resp. of type $(0,1)$). The *complex gradient* $\operatorname{grad}_C \varphi \in \mathcal{X}^{1,0}(M)$ (resp. its *complex conjugate* $\overline{\operatorname{grad}}_C \varphi \in \mathcal{X}^{0,1}(M)$) for any $\varphi \in \mathcal{F}_R(M)$ will be defined by

(1) $$g(\operatorname{grad}_C \varphi, Z)=Z\cdot\varphi, \qquad \text{for every } Z\in\mathcal{X}^{0,1}(M),$$
$$(\text{resp. } g(\overline{\operatorname{grad}}_C \varphi, Z)=Z\cdot\varphi, \qquad \text{for every } Z\in\mathcal{X}^{1,0}(M)).$$

Then we can introduce the *complex hessian tensor field* $\operatorname{Hess}_C(\varphi)$ (resp. its *complex conjugate* $\overline{\operatorname{Hess}}_C(\varphi)$) of type $(1,1)$ over M by the following:

(2) $$\operatorname{Hess}_C(\varphi): X\to\nabla_X(\operatorname{grad}_C\varphi), \qquad \text{for } X\in\mathcal{X}^{1,0}(M),$$
$$(\text{resp. } \overline{\operatorname{Hess}}_C(\varphi): Y\to\nabla_Y(\overline{\operatorname{grad}}_C\varphi), \qquad \text{for } Y\in\mathcal{X}^{0,1}(M)).$$

We note that $\operatorname{trace}\operatorname{Hess}_C(\varphi)=\Box\varphi$ and $\operatorname{trace}\overline{\operatorname{Hess}}_C(\varphi)=\overline{\Box}\varphi$ in the usual notation. Corresponding to (1) and (2), we are now in a position to introduce the two non-linear partial differential operators D_C and $\bar{D}_C$ of the complex (hessian) determinant type as follows:

(3) $$D_C(\varphi)=\det.\operatorname{Hess}_C(\varphi), \qquad \bar{D}_C(\varphi)=\det.\overline{\operatorname{Hess}}_C(\varphi).$$

Moreover, for the later use, we need to introduce the somewhat modified operators, namely for any real number λ, putting that

$$p_C^{(\lambda)}(\varphi)=\operatorname{Hess}_C(\varphi)+\lambda\varphi\cdot I_n \qquad (\text{resp. } \bar{p}_C^{(\lambda)}(\varphi)=\overline{\operatorname{Hess}}_C(\varphi)+\lambda\varphi I_n),$$

where I_n designates the identity operator in $\mathcal{X}(M)$, we define

(4) $$D_C^{(\lambda)}(\varphi)=\det.p_C^{(\lambda)}(\varphi), \qquad \bar{D}_C^{(\lambda)}(\varphi)=\det.\bar{p}_C^{(\lambda)}(\varphi),$$

for any $\varphi\in\mathcal{F}(M)$. We would now like to call these differential operators as the *generalized complex Monge-Ampère operators*.

3. We will be now concerned with the partial differential equations on $M=P_C^n$ with Fubini-Study metric g:

$(5)_\lambda$ $\qquad D_C^{(\lambda)}(\varphi)=\kappa$, or, what is the same $\bar{D}_C^{(\lambda)}(\varphi)=\kappa$,

under the assumption that $\kappa\in\mathcal{F}_R(M)$ is *positive everywhere* and the solutions are limited to the *elliptic* ones in the sense of the preceding note, namely we consider only the case where $p^{(\lambda)}(\varphi)$ (resp. $\bar{p}^{(\lambda)}(\varphi)$) are positive definite hermitian operators with respect to g. As for the equations (5), in a similar way as in [1] we get the following:

Theorem 1. *When $\lambda=1$, the equation $(5)_1$ has the unique elliptic solution φ for any given positive function κ.*

Reference

[1] M. Ise: A complex analogue of the generalized Minkowski problem. I. Proc. Japan Acad., **53A** (4), 129–133 (1977).