60. *Remarks on the Preservation of Uniform Stability from a Linear System to its Perturbed System*

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1. Introduction. We consider the behavior of the solutions of the non-autonomous linear differential system

(L) $$x'=A(t)x$$

and its perturbed system

(PL) $$y'=A(t)y+h(t).$$

The purpose of this paper is to give some necessary conditions and sufficient conditions on the preservation, actually the eventualization (cf. Definitions 2.1 and 2.2), of

(1) uniform stability

(2) uniform stability and attraction

from (L) to (PL).

Strauss-Yorke [1] gave results for this perturbation problem. In this paper we show some more detailed results (Theorems 3.1 and 4.1).

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2. Notations and definitions. We use the following notations throughout this paper. R^n denotes the Euclidean n-space and 0 denotes the origin of R^n. For $x=(x_1, \cdots, x_n)\in R^n$, let $|x|=\sum_{i=1}^n |x_i|$. For an $n\times n$ matrix A, let $|A|=\sup_{x\neq 0}|Ax|/|x|$. $A(t)$ denotes an $n\times n$ matrix valued continuous function and $h(t)$ an n-vector valued continuous function on $[0, \infty)$. $\mathcal{F}_C$ is the class of $n\times n$ matrix valued continuous functions on $[0, \infty)$. $x(t, t_0, x_0)$ (resp. $y(t, t_0, x_0)$) denotes the unique solution of (L) (resp. of (PL)) through (t_0, x_0).

We next present the definitions of stabilities to be used.

Definition 2.1. The origin is *eventually uniform stable* (EvUS) if for every $\varepsilon>0$, there exist $\alpha=\alpha(\varepsilon)\geqq 0$ and $\delta=\delta(\varepsilon)>0$ such that

$$|y(t, t_0, x_0)|<\varepsilon \qquad \text{for all } |x_0|<\delta \text{ and } t\geqq t_0\geqq \alpha.$$

It is *uniform stable* (US) if one can choose $\alpha(\varepsilon)\equiv 0$.

Definition 2.2. The origin is *eventually attracting* (EvAt) if there exist $\delta_0>0$ and $\alpha_0\geqq 0$ such that

$$\lim_{t\to\infty}|y(t, t_0, x_0)|=0 \qquad \text{for each } |x_0|<\delta_0 \text{ and } t_0\geqq \alpha_0.$$

It is *attracting* (At) if one can choose $\alpha_0=0$.

The following definition is due to Strauss-Yorke [1], which is one formulation of a perturbation problem.

Definition 2.3. Let $\mathcal{F}\subset\mathcal{F}_C$. Let Z denote either "US" or "US and At". Let EvZ denote either "EvUS" or "EvUS and EvAt". Define the perturbation class $\mathcal{H}(\mathcal{F})$ of the functions $h(t)$ in (PL) by $\mathcal{H}(\mathcal{F})=\{h: A(t)\in\mathcal{F}$ and "0 is Z for (L)" imply "0 is EvZ for (PL)$\}$".

One can easily show the following

Lemma 2.4. *Let* $\mathcal{F}_1\subset\mathcal{F}_2\subset\mathcal{F}_C$. *Then* $\mathcal{H}(\mathcal{F}_1)\supset\mathcal{H}(\mathcal{F}_2)$.

We consider the following six classes of $A(t)$ of (L).

$\mathcal{F}_B=\{A(t)$: there exists $M>0$ such that $|A(t)|\leqq M$ for $t\geqq 0\}$,

$$\mathcal{F}_1=\left\{A(t): \text{there exists } M>0 \text{ such that } \int_t^{t+1}|A(\tau)|\,d\tau\leqq M \text{ for } t\geqq 0\right\},$$

$$\tilde{\mathcal{F}}_1=\left\{A(t): \text{there exists } M>0 \text{ such that } e^{-t}\int_0^t e^{\tau}|A(\tau)|\,d\tau\leqq M \text{ for } t\geqq 0\right\},$$

$$\mathcal{F}_2=\left\{A(t): \text{there exists } M>0 \text{ such that } \left|\int_t^{t+1}A(\tau)d\tau\right|\leqq M \text{ for } t\geqq 0\right\},$$

$$\tilde{\mathcal{F}}_2=\left\{A(t): \text{there exists } M>0 \text{ such that } \left|e^{-t}\int_0^t e^{\tau}A(\tau)d\tau\right|\leqq M \text{ for } t\geqq 0\right\},$$

$\mathcal{F}_C=\{A(t): A(t)$ is continuous for $t\geqq 0\}$.

Concerning $h(t)$ of (PL), we shall consider the following four classes. Here $\gamma(t)$ is the function defined by $\gamma(t)=e^{-t}\int_0^t e^s h(s)ds$.

$$H_0=\left\{h(t): \int_0^\infty |h(t)|\,dt<\infty\right\},$$

$$H_{\gamma_0}=\left\{h(t): \lim_{t\to\infty}|\gamma(t)|=0 \text{ and } \int_0^\infty |\gamma(t)|\,dt<\infty\right\},$$

$$H_\gamma=\left\{h(t): \lim_{t\to\infty}|\gamma(t)|=0 \text{ and } \lim_{T\to\infty}\int_0^T \gamma(t)dt \text{ exists}\right\},$$

$$H=\left\{h(t): \lim_{T\to\infty}\int_0^T h(t)dt \text{ exists}\right\}.$$

It is easy to show the following inclusions.

(2.1) $$\mathcal{F}_B\subset\mathcal{F}_1=\tilde{\mathcal{F}}_1\subset\tilde{\mathcal{F}}_2\subset\mathcal{F}_2\subset\mathcal{F}_C.$$

Then the next inclusions follow from Lemma 2.4.

(2.2) $$\mathcal{H}(\mathcal{F}_C)\subset\mathcal{H}(\mathcal{F}_2)\subset\mathcal{H}(\tilde{\mathcal{F}}_2)\subset\mathcal{H}(\mathcal{F}_1)\subset\mathcal{H}(\mathcal{F}_B).$$

The following inclusions are also clear.

(2.3) $$H_0\subsetneqq H_{\gamma_0}\subsetneqq H_\gamma\subset H.$$

The strict inclusions in (2.3) are afforded by the following examples. For the first one, let $h(t)=\cos e^t$, then $h(t)\in H_{\gamma_0}$ but $\notin H_0$. For the second one, let $h(t)=\frac{\cos(t+1)-\sin(t+1)}{t+1}-\frac{\cos(t+1)}{(t+1)^2}$, then $h(t)\in H_\gamma$ but $\notin H_{\gamma_0}$.

3. Uniform stability. For the eventualization of uniform stability, Strauss-Yorke [1] have proved the following result.

Theorm A. *For uniform stability and for the classes* $\mathcal{F}_B$ *and* $\mathcal{F}_C$,

$$H_0=\mathcal{H}(\mathcal{F}_C)\subsetneqq\mathcal{H}(\mathcal{F}_B)\subset H. \tag{3.1}$$

This result can be made precise in the following way. An outline of the proof will be given in § 5.

Theorem 3.1. *For uniform stability and for the classes* $\mathcal{F}_B, \mathcal{F}_1, \tilde{\mathcal{F}}_2, \mathcal{F}_2$ *and* $\mathcal{F}_C$, *we have*

$$H_0=\mathcal{H}(\mathcal{F}_C)=\mathcal{H}(\mathcal{F}_2)=\mathcal{H}(\tilde{\mathcal{F}}_2)\subsetneqq\mathcal{H}(\mathcal{F}_1)\subset\mathcal{H}(\mathcal{F}_B)\subset H, \tag{3.2}$$

and

$$H_0\subsetneqq H_{\gamma_0}\subset\mathcal{H}(\mathcal{F}_B)\subset H_\gamma\subset H. \tag{3.3}$$

4. Uniform stability and attraction. The classes of functions $\mathcal{H}(\mathcal{F})$ here in question are those concerned with "EvUS and EvAt". To make explicit the difference with those in the statement in § 3, we denote these perturbation classes by

$$\mathcal{H}^*(\mathcal{F}_B),\ \mathcal{H}^*(\mathcal{F}_1),\ \mathcal{H}^*(\tilde{\mathcal{F}}_2),\ \mathcal{H}^*(\mathcal{F}_2)\quad\text{and}\quad\mathcal{H}^*(\mathcal{F}_C)$$

instead of $\mathcal{H}(\mathcal{F}_B)$, etc. For the eventualization of uniform stability and attraction Strauss-Yorke [1] have proved the following result.

Theorem B. *For uniform stability and attraction, and for the classes* $\mathcal{F}_B$ *and* $\mathcal{F}_C$,

$$H_0=\mathcal{H}^*(\mathcal{F}_C)\subsetneqq\mathcal{H}^*(\mathcal{F}_B)\subset H.$$

Concerning this theorem, we are also able to give a refinement.

Theorem 4.1. *For uniform stability and attraction, and for the classes* $\mathcal{F}_B, \mathcal{F}_1, \tilde{\mathcal{F}}_2, \mathcal{F}_2$ *and* $\mathcal{F}_C$, *we have*

$$H_0=\mathcal{H}^*(\mathcal{F}_C)=\mathcal{H}^*(\mathcal{F}_2)=\mathcal{H}^*(\tilde{\mathcal{F}}_2)\subsetneqq\mathcal{H}^*(\mathcal{F}_1)\subset\mathcal{H}^*(\mathcal{F}_B)\subset H,$$

and

$$H_0\subsetneqq H_{\gamma_0}\subset\mathcal{H}^*(\mathcal{F}_B)\subset H_\gamma\subset H.$$

5. Proof. Let $X(t)$ be a fundamental matrix for (L), then the next lemma is well-known [3, p. 54].

Lemma 5.1. *The origin is uniformly stable for* (L), *if and only if, for some* $K\geqq 1$,

$$|X(t)X^{-1}(s)|\leqq K\qquad \textit{for all } t\geqq s\geqq 0.$$

It is attracting if and only if

$$|X(t)|\to 0\quad \textit{as}\quad t\to\infty.$$

Now, the solution $y(t, t_0, x_0)$ of (PL) is given by

$$y(t, t_0, x_0)=X(t)X^{-1}(t_0)x_0+X(t)\int_{t_0}^{t}X^{-1}(s)h(s)ds. \tag{5.1}$$

If $h^*(t)=-\int_t^\infty h(s)ds$ is defined for some and hence for all $t\geqq 0$, an integration by parts gives

$$(5.2)\qquad \begin{aligned} y(t, t_0, x_0) &= X(t)X^{-1}(t_0)[x_0 - h^*(t_0)] \\ &\quad + h^*(t) + X(t)\int_{t_0}^{t} X^{-1}(s)A(s)h^*(s)ds. \end{aligned}$$

Let $\gamma(t) = e^{-t}\int_0^t e^s h(s)ds$, then we have

$$(5.3)\qquad \begin{aligned} y(t, t_0, x_0) &= X(t)X^{-1}(t_0)[x_0 - \gamma(t_0)] \\ &\quad + \gamma(t) + X(t)\int_{t_0}^{t} X^{-1}(s)[A(s)+I]\gamma(s)ds, \end{aligned}$$

where I denotes the $n\times n$ identity matrix.

Proof of Theorem 3.1. To prove that

$$(5.4)\qquad H_0 = \mathcal{H}(\mathcal{F}_C) = \mathcal{H}(\mathcal{F}_2) = \mathcal{H}(\tilde{\mathcal{F}}_2),$$

it suffices to show that $H_0 \supset \mathcal{H}(\tilde{\mathcal{F}}_2)$ according to (2.2) and (3.1), and this can be done by using the same technique as in [1].

We next show that

$$(5.5)\qquad H_0 \subsetneqq \mathcal{H}(\mathcal{F}_1).$$

To this end, we exhibit an example. For simplicity we consider the scalar case. Let $h(t) = \frac{1}{2}\cos e^t$, then $h(t) \notin H_0$. Suppose that the origin is (US) for (L) for an $A(t) \in \mathcal{F}_1$. Since $\lim_{T\to\infty}\int_0^T h(t)dt$ exists, $h^*(t) = -\int_t^\infty h(s)ds$ is well defined. It follows then from (5.2) and Lemma 5.1 that

$$\begin{aligned} |y(t, t_0, x_0)| &\leqq K(|x_0| + |h^*(t_0)|) + |h^*(t)| \\ &\quad + K\int_{t_0}^{t} |A(s)||h^*(s)|\, ds. \end{aligned}$$

One can show that $|h^*(t)| \leqq e^{-t}$ for all $t \geqq 0$ and there exists $\tilde{M} > 0$ such that $e^{-t}\int_0^t e^s |A(s)|\, ds \leqq \tilde{M}$ for all $t \geqq 0$. Then we have

$$|y(t, t_0, x_0)| \leqq K[|x_0| + (1+\tilde{M})e^{-t_0}] + e^{-t}.$$

Hence the origin is (EvUS) for (PL). We thus have $h(t) \in \mathcal{H}(\mathcal{F}_1)$.

Other relations in (3.2) are immediate from (2.1) and (3.1).

We next prove that

$$(5.6)\qquad H_{r_0} \subset \mathcal{H}(\mathcal{F}_B).$$

Suppose that the origin is (US) for (L) for an $A(t) \in \mathcal{F}_B$ and let $h(t) \in H_{r_0}$. There exists $M^* > 0$ such that $|A(t) + I| \leqq M^*$ for all $t \geqq 0$. It follows from (5.3) and Lemma 5.1 that

$$\begin{aligned} |y(t, t_0, x_0)| &\leqq K(|x_0| + |\gamma(t_0)|) + |\gamma(t)| \\ &\quad + KM^*\int_{t_0}^{t} |\gamma(s)|\, ds. \end{aligned}$$

Since $h(t) \in H_{r_0}$, $\lim_{t\to\infty}|\gamma(t)| = 0$ and $\int_0^\infty |\gamma(t)|\, dt < \infty$. Hence the origin is (EvUS) for (PL). Therefore $h(t) \in \mathcal{H}(\mathcal{F}_B)$.

Finally we prove that

(5.7) $$\mathcal{H}(\mathcal{F}_B) \subset H_\gamma.$$

It suffices to show that for any $h(t) \notin H_\gamma$, $h(t) \notin \mathcal{H}(\mathcal{F}_B)$.

For an $h(t) \notin H_\gamma$, let $\gamma(t) = e^{-t}\int_0^t e^s h(s)ds$. We have two cases:

(i) $\lim_{t\to\infty} |\gamma(t)| \neq 0$,

(ii) $\lim_{t\to\infty} |\gamma(t)| = 0$ and $\lim_{T\to\infty}\int_0^T \gamma(t)dt$ does not exist.

Case (i). Let $A(t) = -1$. We consider

(5.8) $$x' = A(t)x$$

(5.9) $$y' = A(t)y + h(t).$$

Here $A(t) \in \mathcal{F}_B$ and the origin is (US) for (5.8), while the origin is not (EvUS) for (5.9). Hence $h(t) \notin \mathcal{H}(\mathcal{F}_B)$.

Case (ii). In this case, we use the following

(5.10) $$x' = 0$$

(5.11) $$y' = h(t).$$

We thus have sketched the proof of Theorem 3.1.

The proof of Theorem 4.1 is carried out in a similar way. The point here is to check whether the assertion concerning the eventual attraction can be established at each place.

Remark 5.2. The authors have not obtained the complete characterization of the classes $\mathcal{H}(\mathcal{F}_B)$ in Theorem 3.1 and $\mathcal{H}^*(\mathcal{F}_B)$ in Theorem 4.1.

References

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