58. *Holomorphic Continuation of Solutions of Degenerate Partial Differential Equations in Two Variables*

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1. Introduction. Holomorphic continuation of solutions of partial differential equations in the complex domain has been the recent investigations. The main results already obtained to this subject are the continuation theorems which hold across any non-characteristic hypersurface or simply characteristic hypersurface. The purpose of this note is to extend these results to the case where the operator $P(z,\partial_z)$ is highly degenerated at the boundary point.

Let $P(z,\partial_z)$ be a linear partial differential operator of order m with holomorphic coefficients defined near a point p and Ω be an open set such that the point p belongs to the C^2-boundary $\partial\Omega$. We employ here the weighted coordinates introduced by T. Bloom-I. Graham [1], which means that the holomorphic normal direction of $\partial\Omega$ at p is assigned the weight 2, while the tangential directions are each assigned the weight 1. The conditions on P which guarantee the continuation theorem will be made on the "weighted" principal part of P defined in the next section. In this note we shall report the results in the two dimensional cases. The complete proofs of our results and the generalization to the many variable cases will be made somewhere else.

2. Weighted coordinates. Let (z_1, z_2) be a local coordinates. Then we say that (z_1, z_2) is the weighted coordinates with the weight $(2,1)$ if the coordinate function z_1 has the weight 2, while z_2 has the weight 1. A monomial in z has the weight l if the sum of the weight z_1 and z_2 which occur is l. A holomorphic function $f(z)$ has the weight l if, among the monomials in the Taylor series expansion of $f(z)$, there is one of weight l but none of lower weight. For convenience, the weight of $f=0$ is assigned $+\infty$. For differential operators the corresponding negative weights are assigned. The weight of $(\partial/\partial z_1)^j(\partial/\partial z_2)^k$ is $-2j-k$ and the weight of $a(z)(\partial/\partial z_1)^j(\partial/\partial z_2)^k$ is defined by weight$(a(z))$ $+$weight$((\partial/\partial z_1)^j(\partial/\partial z_2)^k)$. Lastly the weight of a linear partial differential operator $P(z,\partial_z)=\sum_{|\alpha|\leq m} a_\alpha(z)(\partial/\partial z)^\alpha$ is, by definition, min weight $(a_\alpha(z)(\partial/\partial z)^\alpha)$.

Definition 2.1 ([1]). Let (z_1, z_2) and (w_1, w_2) be local coordinates with the same origin. We say that these coordinates are equivalent if the coordinate function w_j has the same weight as z_j as a holomorphic function of z and the converse is also true. This means also that

$$\frac{\partial(w_1, w_2)}{\partial(z_1, z_2)}(0)=\begin{vmatrix} a & 0 \\ b & c \end{vmatrix}\neq 0.$$

Since the weights of functions or differential operators are invariant under the equivalent weighted coordinates transformations, we have the following

Proposition 2.1. *Let $P(z,\partial_z)$ be a differential operator of the weight l and denote by $Q(z,\partial_z)$ the sum of the terms in P with the weight l. Then $Q(z,\partial_z)$ is invariant modulo differential operators of the weight larger than l under the equivalent change of the weighted coordinates.*

Definition 2.2. The sum of the terms in P with the lowest weight is said to be the weighted principal part of P.

Example. If $w_1=z_1$, $w_2=z_1+z_2$, then $\partial/\partial z_1=\partial/\partial w_1+\partial/\partial w_2$ where weight $(\partial/\partial z_1)=$weight$(\partial/\partial w_1)=-2$ while weight$(\partial/\partial w_2)=-1$. Thus the weighted principal part of the above operator is $\partial/\partial z_1$ in the z-coordinates and $\partial/\partial w_1$ in the w-coordinates. These two operators are mutually equal modulo operators of the weight -1.

3. Continuation theorem. Suppose that the operator $P(z,\partial_z)$ is defined near the origin and Ω is given by $\{\rho(z)<0\}$ for some C^2 function ρ with $\rho(0)=0$. We first choose any local coordinates (z_1, z_2) such that the surface $z_1=0$ is tangent to $\partial\Omega$ at 0 and consider this coordinates as the weighted coordinates with the weight (2, 1). Let l ($l\geqq 1$) be the multiplicity of P at $(0,N)$ where $N=(1,0)$ is the holomorphic normal direction of $\partial\Omega$. This means that

(3.1) $\quad P_m(0,N+t\zeta)=p(\zeta)t^l+$higher order terms of t, $p(\zeta)\neq 0$,

where P_m is the principal part of P. By (3.1), the weight of $P_m(0,\partial_z)$ is equal to $l-2m$, so we make the following assumption.

(P. I) the weight of $P(z,\partial_z)$ is equal to $l-2m$.

Definition 3.1. A holomorphic function $\phi(z)$ with $\mathrm{grad}_z\,\phi(0)=N$ is said to be a weighted characteristic function of P if it satisfies

(3.2) $\quad e^{-t\phi(z)}Q(z,\partial_z)e^{t\phi(z)}=0 \qquad$ (mod weight larger than $l-2m$),

where $Q(z,\partial_z)$ is the weighted principal part of P and the complex parameter t is assigned the weight -2.

To find such a function $\phi(z)$, it is sufficient that ϕ is in the form

$$\phi(z)=z_1+az_2.$$

Thus the equation (3.2) is essentially algebraic. Now we assume

(P. II) there exists a weighted characteristic function $\phi(z)$.

We then fix some weighted characteristic function $\phi(z)$ and consider the local coordinates (z_1, z_2) as $\phi(z)=z_1$ and z_2 has the weight 1. Then P is written as

$$P(z,\partial_z)=\sum_{k=0}^{m} p_k(z,\partial/\partial z_2)(\partial/\partial z_1)^k.$$

In this expression the weight of p_k is larger than or equal to $l-2m+2k$ and the order of p_k is smaller than or equal to $m-k$. Thus if we set $Q_k(z, \partial/\partial z_2)$ the sum of the terms in p_k of the weight $l-2m+2k$, then $Q_k=0$ if $k-m>l-2m+2k$, that is, if $k<m-l$. Conversely if $k=m-l$, $Q_{m-l}=a(z)(\partial/\partial z_2)^l$ with $a(0)\neq 0$ because the multiplicity of P_m at $(0, N)$ is equal to l. The last assumption on P is the following

(P. III) $Q_k=0$ if $k>m-l$.

The condition (P. III) becomes invariant under the change of variables which fix the z_1-axis, that is, the transformation of the form

$$w_1=z_1$$
$$w_2=az_1+bz_2+f(z), \qquad b\neq 0,\ f=0(|z|^2).$$

Remark. $Q_{m-l}(0, \xi_2)$ is just the localization of $P_m(z, \xi)$ at $(0, N)$ which is due to Hörmander. Indeed the relation

$$P_m(0, N+t\xi)=Q_{m-l}(0, \xi_2)t^l+\text{higher order terms of } t$$

is easy to prove.

The assumption on $\partial\Omega$ is the following

(Ω. I) there exists a non-singular holomorphic curve $\zeta(t)$ with $\zeta(0)=0$ in the weighted characteristic surface $z_1=0$ and a non-zero complex number t_0 such that

$$\frac{d^2}{d\tau^2}(\rho(\zeta(t_0\tau)))\Big|_{\tau=0}<0$$

for a real parameter τ.

Theorem. *Under the assumptions* (P. I), (P. II), (P. III) *and* (Ω. I), *any holomorphic solution $u(z)$ in Ω of $Pu=f$, where f is holomorphic near* 0, *can be holomorphically prolonged across $\partial\Omega$ at* 0.

4. Outline of the proof. By the assumptions in the theorem, we can choose the local coordinates (z_1, z_2) such that P is written as

$$P(z, \partial_z)=\left(\frac{\partial}{\partial z_1}\right)^{m-l}\left(\frac{\partial}{\partial z_2}\right)^{l}+\sum_\alpha a_\alpha(z)\left(\frac{\partial}{\partial z}\right)^{\alpha},$$

where weight$(a_\alpha(z)(\partial/\partial z)^\alpha)$ is larger than $l-2m$, and the second order derivative of ρ, the defining function of Ω, with respect to $\operatorname{Re} z_2$ is negative. Then we seek the discs Δ_1 and Δ_2 which are contained in Ω such that $\Delta_1\subset\{z_1=\text{const. } \delta^2\}$ and $\Delta_2\subset\{z_2=\delta\}$. In this situation we consider $u(z)$ as the solution of the Goursat problem $Pu=f$, with the boundary data on Δ_1 and Δ_2. By the quantitative estimate of the existence-domain of u (which is given by Hörmander [2, Theorem 5.1.1]), u is holomorphic near the origin if we take δ sufficiently small.

Remark. If P is simple characteristic at $(0, N)$, then it is possible to choose the local coordinates (z_1, z_2) such that

$$P=\left(\frac{\partial}{\partial z_1}\right)^{m-1}\left(\frac{\partial}{\partial z_2}\right)+\text{lower order terms of } \partial/\partial z_1,$$

where z_2-axis is the bicharacteristic direction. In this case the weight of P is equal to $1-2m$ and the conditions (P. I), (P. II) and (P. III) are automatically satisfied. Therefore our theorem is just the generalization of the result (Corollary 1 of Theorem 1) in the preceding paper [3].

References

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[3] Y. Tsuno: On the prolongation of local holomorphic solutions of partial differential equations. J. Math. Soc. Japan, **26**, 523–548 (1974).