57. *Studies on Holonomic Quantum Fields. VIII*

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The purpose of this note along with [2] is to extend our previous work [1] concerning a monodromy preserving deformation for solutions of the 2-dimensional Euclidean Dirac equations. Generalization consists in two respects, namely (i) extension of the exponent $l_\nu=0$ of local monodromy to $-1/2\leqq l_\nu\leqq 1/2$, and (ii) introduction of an $n(n-1)/2$ parameter family of global monodromy structures. Construction of the relevant operator theory has been accomplished in the preceding note VII [2]. Here we shall deal with the mathematical part. We show in §§1–2 the existence and uniqueness of wave functions, derive in §3 holonomic systems and its deformation equations and establish in §4 the connection with the operator theory. Detailed discussion will be published in [3].

Unless otherwise stated we shall maintain the same notations used throughout this series [1], [2], [5].

1. Let $\{(a_\nu, \bar{a}_\nu)\}_{\nu=1,\cdots,n}$ be distinct n-points of X^{Euc}. Denote by $\tilde{X}'=\tilde{X}'_{a_1,\cdots,a_n}$ the universal covering manifold of $X'=X'_{a_1,\cdots,a_n}=X^{\text{Euc}}-\{(a_\nu,\bar{a}_\nu)\}_{\nu=1,\cdots,n}$. For $l_1,\cdots,l_n\in\boldsymbol{R}$, let

$$(1)\qquad \rho_{l_1,\cdots,l_n}:\pi_1(X';x_0)\to U(1),\qquad \gamma_\nu\mapsto e^{-2\pi i l_\nu}\ (\nu=1,\cdots,n)$$

be a unitary representation of the fundamental group. Here γ_ν denotes a closed loop with the base point $x_0\in X'$, encircling $(a_\nu,\bar{a}_\nu)$ clockwise. Changing the notation from VII we denote by γu the analytic continuation of a real analytic function u along the path γ^{-1}.

Assume $l_\nu\notin\boldsymbol{Z}$ (resp. $l_\nu\notin\boldsymbol{Z}+1/2$) for $\nu=1,\cdots,n$. We consider the space $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ (resp. $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$) consisting of real analytic functions v (resp. $w={}^t(w_+,w_-)$) on $\tilde{X}'$ satisfying the properties $(2)_B$ (resp. $(2)_F$), (3), $(4)_\nu$ $(\nu=1,\cdots,n)$ and $(4)_\infty$ below:

$$(2)_B\qquad (m^2-\partial_z\partial_{\bar{z}})v=0\quad \text{on } \tilde{X}'$$

$$((2)_F\qquad (m-\Gamma)w=0 \text{ on } \tilde{X}'),$$

$$(3)\qquad \gamma v=v\cdot\rho_{l_1,\cdots,l_n}(\gamma)\qquad (\text{resp. } \gamma w=w\cdot\rho_{l_1-1/2,\cdots,l_n-1/2}(\gamma))\quad \text{for any } \gamma\in\pi_1(X';x_0),$$

$$(4)_\nu\qquad |v|,|\partial_{\bar{z}}v|=O(|z-a_\nu|^{-[l_\nu]-1})\qquad (\text{resp. } |w_\pm|=O(|z-a_\nu|^{-[l_\nu+1/2]-1})\quad \text{as } |z-a_\nu|\to 0,$$

$$(4)_\infty\qquad |v|=O(e^{-2m|z|})\ (\text{resp. } |w_\pm|=O(e^{-2m|z|}))\qquad \text{as } |z|\to\infty.$$

Under the conditions (2) and (3), $(4)_\nu$ is equivalent to

$$(4)'_\nu \qquad v=\sum_{j=0}^{\infty} c_{-l_\nu+j}^{(\nu)}(v)\cdot v_{-l_\nu+j}[a_\nu]+\sum_{j=0}^{\infty} c_{l_\nu^*+j}^{*(\nu)}(v)\cdot v_{l_\nu^*+j}^{*}[a_\nu]$$
$$(\text{resp. } w=\sum_{j=0}^{\infty} c_{-l_\nu+j}^{(\nu)}(w)\cdot w_{-l_\nu+j}[a_\nu]+\sum_{j=0}^{\infty} c_{l_\nu^*+j}^{*(\nu)}(w)\cdot w_{l_\nu^*+j}^{*}[a_\nu])$$
$$c_{-l_\nu+j}^{(\nu)}(v),\ c_{l_\nu^*+j}^{*(\nu)}(v),\ c_{-l_\nu+j}^{(\nu)}(w),\ c_{l_\nu^*+j}^{*(\nu)}(w)\in \boldsymbol{C},$$

where $l_\nu^*=l_\nu-2[l_\nu]$ (resp. $l_\nu^*=l_\nu-2[l_\nu+1/2]$) and v_l, v_l^* (resp. w_l, w_l^*) denote local solutions of (2) introduced in II–(2) [1]. By the definition we have

$$(5) \qquad W_{B,a_1,\cdots,a_n}^{l_1+1/2,\cdots,l_n+1/2} \rightleftarrows W_{F,a_1,\cdots,a_n}^{l_1,\cdots,l_n}, \qquad v\mapsto {}^t(v, m^{-1}\partial_{\bar z}v).$$

If $0<l_\nu<1$ (resp. $-1/2<l_\nu<1/2$), $\nu=1,\cdots,n$, a positive definite hermitian inner product is introduced by setting

$$(6) \qquad I_B(v,v')=\frac{1}{2}\iint_{X^{\mathrm{Euc}}} idz\wedge d\bar z(\partial_{\bar z}v\cdot\partial_z\bar v'+m^2 v\bar v'),$$
$$\left(\text{resp. } I_F(w,w')=\frac{m^2}{2}\iint_{X^{\mathrm{Euc}}} idz\wedge d\bar z(w_+\bar w'_+ + w_-\bar w'_-)\right)$$

for $w, w'\in W_{B,a_1,\cdots,a_n}^{l_1,\cdots,l_n}$ (resp. $W_{F,a_1,\cdots,a_n}^{l_1,\cdots,l_n}$), where the integrand is single-valued by virtue of (3). We find

$$(7) \qquad I_B(v,v')=-\sum_{\nu=1}^{n} c_{-l_\nu}^{(\nu)}(v)\overline{c_{l_\nu}^{*(\nu)}(v')}\cdot\sin\pi l_\nu$$
$$(\text{resp. } I_F(w,w')=-\sum_{\nu=1}^{n} c_{-l_\nu}^{(\nu)}(w)\cdot\overline{c_{l_\nu}^{*(\nu)}(w')}\cdot\cos\pi l_\nu).$$

Results in II–§ 2 [1] are generalized as follows.

Theorem 1. *Assume* $0<l_\nu<1$ *for* $*=B$ *and* $-1/2<l_\nu<1/2$ *for* $*=F$, $\nu=1,\cdots,n$. *Then* $\dim_{\boldsymbol{C}} W_{*,a_1,\cdots,a_n}^{l_1,\cdots,l_n}=n$ $(*=B,F)$. *There exists a canonical basis* $v_\mu(L)=v_\mu(z,\bar z;L)$ *or* $w_\mu(L)=w_\mu(z,\bar z;L)$ $(\mu=1,\cdots,n;\ L=(\delta_{\mu\nu}l_\nu)_{\mu,\nu=1,\cdots,n})$ *characterized by the condition*

$$(8)_B \qquad c_{-l_\nu}^{(\nu)}(v_\mu)=\delta_{\mu\nu} \qquad (\mu,\nu=1,\cdots,n)$$
$$(8)_F \qquad c_{-l_\nu}^{(\nu)}(w_\mu)=\delta_{\mu\nu} \qquad (\mu,\nu=1,\cdots,n).$$

Setting $c_{-l_\nu+1}^{(\nu)}(v_\mu)=\alpha_{\mu\nu}(L)$ *and* $c_{l_\nu}^{*(\nu)}(v_\mu)=\beta_{\mu\nu}(L)$, *we have* $c_{-l_\nu+1}^{(\nu)}(w_\mu)=\alpha_{\mu\nu}(L+1/2)$, $c_{l_\nu}^{*(\nu)}(w_\mu)=\beta_{\mu\nu}(L+1/2)$.

Theorem 2. *Notations being as above,* $\bigcup_{j=0}^{\infty} W_{*,a_1,\cdots,a_n}^{l_1+j,\cdots,l_n+j}$ *is a left* $\boldsymbol{C}[\partial_z,\partial_{\bar z},M_*]$*-module* $\left(*=B,F;\ M_B=z\partial_z-\bar z\partial_{\bar z},\ M_F=M_B+\frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\right)$. *We have for* $*=B$ *or* F

$$(9)_* \qquad (\boldsymbol{C}[\partial_z,\partial_{\bar z}]/(m^2-\partial_z\partial_{\bar z}))_j\otimes_{\boldsymbol{C}} W_{*,a_1,\cdots,a_n}^{l_1,\cdots,l_n} \rightleftarrows W_{*,a_1,\cdots,a_n}^{l_1+j,\cdots,l_n+j}$$

by the map $p(\partial_z,\partial_{\bar z})\otimes w\mapsto p(\partial_z,\partial_{\bar z})w$, *where* $(\boldsymbol{C}[\partial_z,\partial_{\bar z}]/(m^2-\partial_z\partial_{\bar z}))_j=\{p(\partial_z,\partial_{\bar z})=\sum_{k=0}^{j} c_k(m^{-1}\partial_z)^k+\sum_{k=1}^{j} c_{-k}(m^{-1}\partial_{\bar z})^k,\ c_k\in\boldsymbol{C}\}$.

The above arguments are generalized to include integral or half integral exponents l_ν. In particular, for given $(z^*,\bar z^*)\in X^{\mathrm{Euc}}-\{(a_\nu,\bar a_\nu)\}_{\nu=1,\cdots,n}$ we can show the existence of a solution $v_0=v_0(L)$ of $(2)_B$, satisfying (3), $(4)_\nu$ with $c_{-l_\nu}^{(\nu)}(v_0)=0$ $(v=1,\cdots,n)$, $(4)_\infty$ and in addition

$$(4)_{B,0} \qquad v_0(L)=\tilde v_0[z^*]+\text{regular function}$$

at $(z^*,\bar z^*)$. Here we have set $\tilde v_l(re^{i\theta}/2, re^{-i\theta}/2)=e^{il(\theta+\pi)}K_l(mr)$. Likewise there exist solutions $w_0^{(\pm)}=w_0^{(\pm)}(L)$ of $(2)_F$ satisfying (3), $(4)_\nu$ with $c_{-l_\nu}^{(\nu)}(w_0^{(\pm)})=0$ $(\nu=1,\cdots,n)$, $(4)_\infty$ and

$$(4)_{F,0} \qquad w_0^{(+)}(L)=-\tilde w_{1/2}^*[z^*]+\text{regular function}$$

$$w_0^{(-)}(L) = \tilde{w}_{1/2}[z^*] + \text{regular function}$$

at $(z^*, \bar{z}^*)$, where $\tilde{w}_l = {}^t(\tilde{v}_{l-1/2}, \tilde{v}_{l+1/2})$ and $\tilde{w}_l^* = {}^t(\tilde{v}_{-l-1/2}, \tilde{v}_{-l+1/2}) = \tilde{w}_{-l}$.

2. In the sequel we assume $-1/2 < l_1, \cdots, l_n < 1/2$. Let $\boldsymbol{w}(L) = {}^t({}^t w_1(L), \cdots, {}^t w_n(L))$ denote the column vector of length $2n$ formed by the canonical basis of $W^{l_1, \cdots, l_n}_{F, a_1, \cdots, a_n}$. It is shown that $\boldsymbol{w}(L)$ depends analytically on the parameters $\{(a_\nu, \bar{a}_\nu)\}_{\nu=1,\cdots,n}$ as long as they are mutually distinct. We set $A = (\delta_{\mu\nu} a_\nu)$, $L = (\delta_{\mu\nu} l_\nu)$. By virtue of Theorems 1 and 2, we have the following results.

Theorem 3. *The vector $\boldsymbol{w} = \boldsymbol{w}(L)$ satisfies the following holonomic system of linear differential equations in the total set of variables $(z, \bar{z}, A, \bar{A})$:*

$$(m - \Gamma)\boldsymbol{w} = 0$$

$$M_F \boldsymbol{w} = (A\partial_z - G^{-1}\bar{A}G\partial_{\bar{z}} + F)\boldsymbol{w} \tag{10}$$

$$d_{A,\bar{A}}\boldsymbol{w} = (-dA \cdot \partial_z - G^{-1}d\bar{A} \cdot G\partial_{\bar{z}} + \Theta)\boldsymbol{w} \tag{11}$$

($d_{A,\bar{A}}$: *exterior differentiation with respect to* $(A, \bar{A})$).

Here F, G and Θ denote $n \times n$ matrices of 0- and 1-forms in $(A, \bar{A})$, respectively, given by

$$F = [\alpha, mA] - L, \qquad G^{-1} = -\beta \cdot \cos \pi L \tag{12}$$

$$\Theta = -[\alpha, m dA],$$

with

$$\alpha = (\alpha_{\mu\nu}(L + 1/2)), \qquad \beta = (\beta_{\mu\nu}(L + 1/2)). \tag{13}$$

Furthermore F and G are subject to the algebraic conditions

$${}^t\bar{F} = GFG^{-1}, \ G = {}^t\bar{G} \text{ is positive definite.} \tag{14}$$

Theorem 4. *The matrices F and G in* (12), (13) *satisfy the following completely integrable system of non-linear total differential equations*

$$dF = [\Theta, F] + m^2([dA, G^{-1}\bar{A}G] + [A, G^{-1}d\bar{A} \cdot G]) \tag{15}$$

$$dG = -G\Theta - \Theta^* G.$$

Here Θ, Θ^* denote matrices of 1-forms characterized by

$$[\Theta, A] + [F, dA] = 0, \text{ diagonal of } \Theta = 0, \tag{16}$$

$$[\Theta^*, \bar{A}] + [GFG^{-1}, d\bar{A}] = 0, \text{ diagonal of } \Theta^* = 0.$$

These results are generalizations of those obtained in II, where the case $L=0$ is discussed. The system (15) ensures the integrability condition for (2), (10), (11).

It is also possible to characterize $w_0 = (w_0^{(+)}(L), w_0^{(-)}(L))$ by means of differential equations. The result is as follows.

$$(m - \Gamma)w_0 = 0$$

$$\begin{cases} (\partial_{z^*} + \sum_{\nu=1}^n \partial_{a_\nu} + \partial_z)w_0 = 0 \\ (\partial_{\bar{z}^*} + \sum_{\nu=1}^n \partial_{\bar{a}_\nu} + \partial_{\bar{z}})w_0 = 0 \\ (M^*_{F,z^*} + \sum_{\nu=1}^n M_{B,a_\nu} + M_{F,z})w_0 = 0 \end{cases} \tag{17}$$

$$(18)\quad \begin{cases} m^{-1}\partial_{a_\nu} w_0 = -\dfrac{\pi}{2\cos \pi l_\nu} w_\nu(z,\bar z\,;L)\cdot {}^t w_\nu(z^*,\bar z^*\,;1-L) \\ m^{-1}\partial_{\bar a_\nu} w_0 = -\dfrac{\pi}{2\cos \pi l_\nu} w_\nu^*(z,\bar z\,;1-L)\cdot {}^t w_\nu^*(z^*,\bar z^*\,;L) \\ \qquad\qquad\qquad\qquad\qquad (\nu=1,\cdots,n). \end{cases}$$

Here we have set $M^*_{F,z^*}w_0=(z^*\partial_{z^*}-\bar z^*\partial_{\bar z^*})w_0+w_0\dfrac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}$, $M_{B,a_\nu}=a_\nu\partial_{a_\nu}-\bar a_\nu\partial_{\bar a_\nu}$.

Notice that these results (2) and (10), (11) or (17), (18) include differential equations for $v_\nu(L)$ or $v_0(L)$, in view of the isomorphism (5).

By the same argument as in II, it follows that if we deform the linear system (2)+(10) in $(z,\bar z)$ according to (11), then the associated monodromy representation is independent of $(A,\bar A)$.

3. Let us now proceed to the connection with the operator theory. We change the definition $\rho_F(x\,;l)$ in VII [2]; namely we replace $R(u,u'\,;l)$ in VII–(4) by

$$R_F(u,u'\,;l)=-2i\cos \pi l\left(\frac{u-i0}{u'-i0}\right)^{-l+1/2}\frac{\sqrt{u-i0}\sqrt{u'-i0}}{u+u'-i0}$$
$$(=R(u,u'\,;l-1/2))$$

and define $\varphi_F(x\,;l)$, $\varphi^F_{l'}(x\,;l)$ and $\varphi^{F*}_{l'}(x\,;l)$ by VII–(5) with the new $\rho_F(x\,;l)$. Let $a_1,\cdots,a_n\in X^{\mathrm{Min}}$ be mutually spacelike points. We set

$$(19)\quad \tau_F(L)w_F^{(\pm)}(x^*,x\,;L)=\pi i\langle\psi^*_\pm(x^*)\varphi_F(a_1\,;l_1)\cdots\varphi_F(a_n\,;l_n)\psi(x)\rangle$$
$$\tau_F(L)w_{F,\nu}(x\,;L)=2i\cos\pi l_\nu\langle\varphi_F(a_1\,;l_1)\cdots\varphi^{F*}_{l_\nu}(a_\nu\,;l_\nu)\cdots\varphi_F(a_n\,;l_n)\psi(x)\rangle$$
$$(\nu=1,\cdots,n)$$

$$(20)\quad \tau_F(L)=\tau_F(a_1,\cdots,a_n\,;L)=\langle\varphi_F(a_1,l_1)\cdots\varphi_F(a_n\,;l_n)\rangle.$$

These functions are analytically prolongable to the domain

$$\mathrm{Im}\,(x^*-a_\nu)^\pm<0,\ \mathrm{Im}\,(a_\mu-a_\nu)^\pm<0\ (\mu<\nu)\quad\text{and}\quad \mathrm{Im}\,(x-a_\nu)^\pm>0$$

in $(X^C)^{n+2}$, in particular to the portion of the Euclidean region $(X^{\mathrm{Euc}})^{n+2}$ defined by these inequalities. Assume as before $-1/2<l_1,\cdots,l_n<1/2$.

Theorem 5. *The Euclidean continuations of* $w_{F,\nu}(x\,;L)$ $(\nu=1,\cdots,n)$ *provide the canonical basis of* $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$. *Likewise* $w_F^{(\pm)}(x^*,x:L)$ *are continued to result in* $w_0^{(\pm)}(L)$ *in* $(4)_{F,0}$. *Hence they are solutions of the holonomic system* (2) *and* (10), (11), *or* (17), (18), *respectively.*

Theorem 6. *The logarithmic derivative* $\omega=d\log\tau_F(L)$ *of the (Euclidean)* τ*-function is given by*

$$(21)\quad \omega=-\frac{1}{2}\,\mathrm{trace}\,(F\Theta+\Theta^*GFG^{-1})$$
$$+m^2\,\mathrm{trace}\,((\bar A-G^{-1}\bar AG)dA+(A-GAG^{-1})d\bar A),$$

where F,G *are the solutions of* (15) *corresponding to* $w=w(L)$.

We remark that the algebraic relations of the type (48)–(50) in II[1], among the various vacuum expectation values involving ψ- and φ-fields, remain valid; indeed they are direct consequences of the product formula (1.4.10) of [4] (see also V [5]). Therefore we have a complete characterization of the wave- and τ-functions of differential equations.

4. Finally we mention a few words on the introduction of the parameter $\Lambda={}^t\Lambda=(\lambda_{\mu\nu})_{\mu,\nu=1,\cdots,n}$. We assume $\lambda_{\nu\nu}=1$ ($\nu=1,\cdots,n$) and that Λ is real, positive definite. In place of (3) we set the following monodromic property for an n-tuple $w=(w^{(1)},\cdots,w^{(n)})$

$$\gamma w = w\cdot\rho_{l_1,\cdots,l_n,\Lambda}(\gamma),\qquad \gamma\in\pi_1(X';x_0) \tag{22}$$

where $\rho_{l_1,\cdots,l_n,\Lambda}(\gamma_\nu)=1+(e^{-2\pi i l_\nu}-1)E_\nu\Lambda$, $E_\nu=(\delta_{\mu\nu}\delta_{\mu'\nu})_{\mu,\mu'=1,\cdots,n}$. Using (22) we define $W^{l_1,\cdots,l_n}_{*,a_1,\cdots,a_n}(\Lambda)$ analogously, where $(4)_\nu$ is to be replaced by

$$(23)_\nu \qquad \begin{aligned} w^{(\mu)} &= \sum_{j=0}^{\infty}\lambda_{\mu\nu}c^{(\nu)}_{-l_\nu+j}(w)\cdot v_{-l_\nu+j}[a_\nu] \\ &\quad+\sum_{j=0}^{\infty}\lambda_{\mu\nu}c^{*(\nu)}_{l_\nu^*+j}(w)\cdot v^*_{l_\nu+j}[a_\nu] \\ &\quad+\text{regular function}\end{aligned}$$

for $*=B$. Modification for $*=F$ is obvious (note that this definition differs from VII–(19) for $|l_\nu|>1/2$). The inner product is defined similarly, with the integrand replaced by the single-valued functions $\partial_{\bar z}v\cdot\Lambda^{-1t}(\partial_z\bar v')+m^2v\Lambda^{-1t}\bar v'$ or $w_+\Lambda^{-1t}\bar w'_+ + w_-\Lambda^{-1t}\bar w'_-$. All the results of §§ 1–3 are generalized to the case of $W^{l_1,\cdots,l_n}_{*a_1,\cdots,a_n}(\Lambda)$ as well. Details will appear in [3].

Errata. IV [1], P. 183, l. 11: $C_{F,l}[A]_l w$ should read $C_{F,l}w_l[A]$.
VII [2], P. 39, l. 2: The definition of M_ν should read

$$M_\nu=1+(e^{2\pi}i^{l_\nu}-1)E_\nu\Lambda.$$

References

[1] M. Sato, T. Miwa, and M. Jimbo: Proc. Japan Acad., **53A**, 147–152, 153–158, 183–185 (1977).
[2] ——: Ibid., **54A**, 36–41 (1978).
[3] ——: Holonomic Quantum Fields. III, RIMS preprint, no. 260; ditto. IV, ibid., no. 263.
[4] ——: Publ. RIMS, **14**, 223–267 (1978).
[5] ——: Proc. Japan Acad., **53A**, 219–224 (1977).