56. *The Hadamard Variational Formula for the Green Functions of Some Normal Elliptic Boundary Value Problems*

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§ 1. **Introduction.** Let Ω be a bounded domain in R^n with the boundary γ of class C^∞. Let $\rho(x)$, $x \in \gamma$, be a real smooth function defined on the boundary γ. For small $\varepsilon \geqq 0$, let Ω_ε be the domain bounded by the hyper-surface $\gamma_\varepsilon = \{y \in \mathrm{R}^n \mid y = x + \varepsilon\rho(x)\nu_x,\ x \in \gamma\}$, where ν_x denotes the unit outer normal vector to γ at $x \in \gamma$. Clearly, $\Omega_0 = \Omega$ and $\gamma_0 = \gamma$. Let $G(\varepsilon, x, y)$ denote the Green function of the Dirichlet boundary value problem for the Laplacian, i.e.,

$$-\Delta G(\varepsilon, x, y) = \delta(x-y), \qquad \text{for } {}^\forall(x, y) \in \Omega_\varepsilon \times \Omega_\varepsilon, \tag{1.1}$$

$$G(\varepsilon, x, y) = 0 \qquad \text{for } x \text{ in } \gamma_\varepsilon \text{ and } y \text{ in } \Omega_\varepsilon. \tag{1.2}$$

We abbreviate $G(0, x, y)$ as $G(x, y)$. Let

$$\delta G(x, y) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(G(\varepsilon, x, y) - G(x, y)) \tag{1.3}$$

for any x and y in Ω. Then, the celebrated Hadamard variational formula reads

$$\delta G(x, y) = \int_\gamma \frac{\partial G(x, z)}{\partial \nu_z} \frac{\partial G(y, z)}{\partial \nu_z} \rho(z) d\sigma(z), \tag{1.4}$$

where $d\sigma(z)$ denotes the surface element of the boundary hyper-surface γ.

Hadamard proved his formula in the case that the function $\rho(z)$ did not change sign. Proof of the formula (1.4) for general $\rho(z)$ can be found, for example, in Garabedian-Schiffer [4], Garabedian [5] and Aomoto*) [2]. A few of new applications of the Hadamard variational formula are found in, for instance, Aomoto [2] and Fujiwara-Tanikawa-Yukita [3]. Since the work of Hadamard, variational formulas are known for the Green functions of certain classical elliptic boundary value problems. For instance, variational formula for the Green function of iterated Laplacian under the Dirichlet boundary condition was given already by Hadamard [6].

The aim of this note is to generalize these and prove variational formulas for the Green functions of some normal elliptic boundary value problems. Our proof is a simple modification of Hadamard's

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original discussions in [6]. What make our proof rigorous are the Whitney extension theorem in [8] of differentiable functions and a priori estimates of Agmon-Douglis-Nirenberg [1].

§2. Statement of results. Let $A(x, D)$ be an elliptic linear partial differential operator of order $2m$ whose coefficients are defined and of class C^∞ in some neighbourhood U of $\bar{\Omega}$. Since Ω_ε is the domain bounded by the hypersurface $\gamma_\varepsilon=\{z+\varepsilon\rho(z)\nu_z \mid z\in\gamma\}$, the closure $\bar{\Omega}_\varepsilon$ of Ω_ε is contained in U if ε is smaller than some positive number ε_0.

We consider elliptic boundary value problem in Ω_ε;

$$\begin{cases} A(x, D)u(x)=f(x) & \text{for } x\in\Omega_\varepsilon, \\ B_j(\varepsilon, x, D)u(x)=0,\ j=1, 2, \cdots, m, & \text{for } x \text{ in } \gamma_\varepsilon, \end{cases} \tag{2.1}$$

where $B_j(\varepsilon, x, D)$ is a boundary differential operator of order m_j and depending on the parameter ε. We assume the following assumptions throughout this paper;

(Ass. 1) The order m_j of the boundary operator $B_j(\varepsilon, x, D)$ is less than $2m$ and independent of ε if ε is small enough.

(Ass. 2) If we represent $B_j(\varepsilon, x, D)$ as

$$B_j(\varepsilon, x, D)=\sum_{|\alpha|\le m_j} b_{j\alpha}(\varepsilon, x)\left(\frac{\partial}{\partial x}\right)^\alpha,$$

then for any α and j, the function $b_{j\alpha}(\varepsilon, x)$ is a C^∞ function of

$$(\varepsilon, x)\in(-\varepsilon_0, \varepsilon_0)\times U.$$

(Ass. 3) $A(x, D)$ is properly elliptic at every point of γ.

(Ass. 4) The system of boundary conditions $\{B_j(0, x, D)\}_{j=1}^m$ is normal in the sense of Schechter [7].

(Ass. 5) The system $\{A(x, D), \{B_j(0, x, D)\}_{j=1}^m\}$ satisfies the complementing condition in the sense of Agmon-Douglis-Nirenberg [1].

(Ass. 6) The boundary value problem (2.1) with $\varepsilon=0$, gives rise to a self-adjoint operator A_0 which is an isomorphism of its domain $D(A_0)=\{u\in H^{2m}(\Omega) \mid B_j(0, x, D)u|_\gamma=0 \text{ for } j=1, \cdots m\}$ onto $L^2(\Omega)$. Here $H^s(\Omega)$ denotes the Sobolev space of order $s\in\mathbf{R}$.

As a simple consequence of the above assumptions, we have

Lemma 2.1. *Assume that assumptions* (Ass. 1)–(Ass. 6) *hold. Then, for every sufficiently small* ε, *the following properties hold:*

(i) *$A(x, D)$ is properly elliptic at every point of γ_ε.*

(ii) *The system of boundary differential operators $\{B_j(\varepsilon, x, D)\}_{j=1}^m$ is normal in the sense of Schechter* [7].

(iii) *The system $\{A(x, D), \{B_j(\varepsilon, x, D)\}_{j=1}^m\}$ satisfies the complementing condition in the sense of Agmon-Douglis-Nirenberg* [1].

Before stating our results, we must recall the discussions in Schechter [7] about the Green's formula. First we take another normal system of boundary differential operators $\{S_j(\varepsilon, x, D)\}_{j=1}^m$ such that $\{B_j(\varepsilon, x, D)\}\cup\{S_j(\varepsilon, x, D)\}$ forms a Dirichlet system in the sense of Schechter, that is, orders of the operators $B_j(\varepsilon, x, D)$ and $S_k(\varepsilon, x, D)$

are all different and fill up the set $\{0, 1, \cdots, 2m-1\}$. Clearly, we can choose $S_j(\varepsilon, x, D)$ so that (Ass. 2) holds with $B_j(\varepsilon, x, D)$ replaced by $S_j(\varepsilon, x, D)$. Second, there exists a unique Dirichlet system $\{B'_j(\varepsilon, x, D\}_{j=1}^m \cup \{S'_j(\varepsilon, x, D)\}_{j=1}^m$ such that we have Green's formula

$$\begin{aligned}(2.3)\qquad &\int_{\Omega^\varepsilon} A(x, D)u(x)\overline{v(x)}dx - \int_{\Omega^\varepsilon} u(x)\overline{A(x, D)^* v(x)}dx \\ &= \sum_j \int_{\gamma_\varepsilon} S_j(\varepsilon, x, D)u(x)\overline{B'_j(\varepsilon, x, D)v(x)}d\sigma^\varepsilon(x) \\ &\quad - \sum_j \int_{\gamma_\varepsilon} B_j(\varepsilon, x, D)u(x)\overline{S'_j(\varepsilon, x, D)v(x)}d\sigma^\varepsilon(x),\end{aligned}$$

for any u and v in $H^{2m}(\Omega_\varepsilon)$. Here $d\sigma^\varepsilon(x)$ denotes the volume element of the hyper-suface γ_ε. (Ass. 2) holds with $B'_j(\varepsilon, x, D)$ and $S'_j(\varepsilon, x, D)$ in place of $B_j(\varepsilon, x, D)$ and $S_j(\varepsilon, x, D)$, respectively. We have two Dirichlet systems

$$\{B_j(\varepsilon, x, D)\}_{j=1}^m \cup \{S_j(\varepsilon, x, D)\}_{j=1}^m \quad \text{and} \quad \{B'_j(\varepsilon, x, D)\}_{j=1}^m \cup \{S'_j(\varepsilon, x, D)\}_{j=1}^m.$$

There is a unique system of linear partial differential operators $\{T_j^k(\varepsilon, x, D)\}_{k=1}^{2m}$ on γ_ε containing only differentiation in the direction tangential to the hyper-surface γ_ε such that

$$(2.4)\quad B'_j(\varepsilon, x, D) = \sum_{k=1}^m T_j^k(\varepsilon, x, D)B_k(\varepsilon, x, D) + \sum_{k=1}^m T_j^{k+m}(\varepsilon, x, D)S_k(\varepsilon, x, D),$$

$$(2.5)\quad S'_j(\varepsilon, x, D) = \sum_{k=1}^m T_{j+m}^k(\varepsilon, x, D)B_k(\varepsilon, x, D) + \sum_{k=1}^m T_{j+m}^{k+m}(\varepsilon, x, D)S_k(\varepsilon, x, D).$$

Lemma 2.2. *Assume that* (Ass. 1)–(Ass. 6) *hold. If the boundary value problem* (2.1) *defines a self-adjoint operator* A_ε, *then*

$$(2.6)\qquad T_k^{j+m}(\varepsilon, x, D) = 0, \qquad \text{for } j, k = 1, 2, \cdots, m.$$

Corollary. *Assume that* (Ass. 1)–(Ass. 6) *are satisfied. Then,*

1°) *We can choose* $B'_j(0, x, D) = B_j(0, x, D)$, $j = 1, 2, \cdots, m$.

2°) *There exists a system of linear partial differential operator* $H_k^{j+m}(\varepsilon, x, D)$ *such that*

$$(2.7)\qquad T_k^{j+m}(\varepsilon, x, D) = \varepsilon H_k^{j+m}(\varepsilon, x, D), \qquad j, k = 1, 2, \cdots, m.$$

The operator $H_k^{j+m}(\varepsilon, x, D)$ *contains only differentiations tangential to* γ_ε.

Now we can state our results. For the sake of brevity, we denote $B_j(0, x, D)$ by $B_j(x, D)$, $S_j(0, x, D)$ by $S_j(x, D)$ and so forth.

Theorem 1. *Assume that* (Ass. 1)–(Ass. 6) *hold. Then,*

1) *For any sufficiently small* $\varepsilon > 0$, *the elliptic boundary value problem*

$$(2.8)\qquad \begin{cases} A(x, D)u(x) = f(x) & \text{for } x \text{ in } \Omega_\varepsilon, \\ B_j(\varepsilon, z, D)u(z) = 0 & \text{for } z \text{ in } \gamma_\varepsilon,\ j = 1, \cdots, m, \end{cases}$$

has a unique solution u *in* $H^{2m}(\Omega_\varepsilon)$ *for any* f *in* $L^2(\Omega_\varepsilon)$.

2) *Let* $G(\varepsilon, x, y)$ *be the Green function for the boundary value problem* (2.8), *i.e.,*

$$(2.9)\quad \begin{cases} A(x, D)G(\varepsilon, x, y) = \delta(x-y) & \text{for any } x \text{ and } y \text{ in } \Omega_\varepsilon, \\ B_j(\varepsilon, z, D)G(\varepsilon, z, y) = 0, & j=1, 2, \cdots, m, \\ & \text{for any } z \text{ in } \gamma_\varepsilon \text{ and } y \text{ in } \Omega_\varepsilon. \end{cases}$$

Then, we have the variational formula

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(G(\varepsilon, x, y) - G(x, y))$$

$$(2.10)\quad \begin{aligned} &= -\sum_{j=1}^{m} \int_\gamma \frac{\partial}{\partial \nu_z}(B_j(z, D)G(z, x))\overline{S'_j(z, D)G(z, y)}\rho(z)d\sigma(z) \\ &\quad + \sum_{j=1}^{m}\sum_{k=1}^{m} \int_\gamma H_j^{k+m}(0, z, D)S_k(z, D)G(z, y)\overline{S_j(z, D)G(z, y)}d\sigma(z) \\ &\quad - \sum_{j=1}^{m} \int_\gamma \left(\frac{\partial}{\partial \varepsilon}B_j(0, z, D)\right)G(z, y)\overline{S'_j(z, D)G(z, y)}d\sigma(z), \end{aligned}$$

where $\frac{\partial}{\partial\varepsilon}B_j(0, z, D) = \sum_{|\alpha|\le m_j} \frac{\partial}{\partial\varepsilon} b_{j\alpha}(\varepsilon, z)\Big|_{\varepsilon=0}\left(\frac{\partial}{\partial z}\right)^\alpha$ *and* $G(x, y) = G(0, x, y)$ *as an abbreviation.*

In particular, the formula (2.10) becomes a little simpler in the following case.

Theorem 2. *Assume that the following* (Ass. 7) *holds as well as* (Ass. 1)–(Ass. 6):

(Ass. 7) *The boundary value problem* (2.1) *defines a self-adjoint operator for each* ε.

Then, we obtain the following formula;

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(G(\varepsilon, x, y) - G(x, y))$$

$$(2.11)\quad \begin{aligned} &= -\sum_{j=1}^{m} \int_\gamma \frac{\partial}{\partial \nu_z}(B_j(z, D)G(z, x))\overline{S'_j(z, D)G(z, y)}\rho(z)d\sigma(z) \\ &\quad - \sum_{j=1}^{m} \int_\gamma \left(\frac{\partial}{\partial \varepsilon}B_j(0, z, D)\right)G(z, x)\overline{S'_j(z, D)G(z, y)}d\sigma(z). \end{aligned}$$

§ 3. Sketch of the proof. Theorem 2 follows from Theorem 1 and Lemma 2.2. Thus we have only to prove Theorem 1. The first part of Theorem 1 is a consequence of a priori estimate of Agmon-Douglis-Nirenberg [1]. We present here a sketch of the proof of the second part of Theorem 1, i.e., the proof of the variational formula (2.10). We fix two points x and y in Ω such that $x \neq y$. Take a compact set $K \subset \Omega$ containing x and y. Consider $G(z, x)$ as a function of z. Then this is defined for z in $\bar{\Omega}$ and not defined for z outside $\bar{\Omega}$. This is the reason why Hadamard assumed that $\rho(z)$ did not change sign. We avoid this difficulty by taking a Whitney extension of $G(z, x)$. Since γ is of class C^∞ and the function $G(z, x)$ is a C^∞ function of z in $\bar{\Omega} - \{x\}$, there exists a Whitney extension $\tilde{G}(z, x)$ of $G(z, x)$ as a function of z. $\tilde{G}(z, x)$ is defined and C^∞ for z in $\mathrm{R}^n - \{x\}$. We have

$$(3.1)\quad A(z, D)\tilde{G}(z, x) = \delta(z-x) + g(z, x),$$

where $g(z, x)$ is a C^∞ function of $z \in \mathbf{R}^n$ and satisfies the equation

$$g(z, x)=0, \qquad \text{for } z \text{ in } \bar{\Omega}. \tag{3.2}$$

In particular, all the derivatives of $g(z, x)$ with respect to z vanish if z is in γ. Let $\zeta=|x-y|$. Let $\varphi(z)$ be a C^∞ function of z such that $\varphi(z)=1$ in a small neighbourhood of x and vanishes if $|z-x|\geqq \frac{1}{2}\zeta$. Let $\langle\ ,\ \rangle$ denote the pairing of $\mathcal{E}'(\Omega_\varepsilon)$ and $\mathcal{E}(\Omega_\varepsilon)$. Then we have

$$\begin{aligned} G(\varepsilon, x, y) &= \langle G(\varepsilon, z, y), \delta(z-x)\rangle \\ &= \langle G(\varepsilon, z, y)\varphi(z), \overline{A(z, D)\tilde{G}(z, x)}\rangle \\ &\quad + \langle G(\varepsilon, z, y), \overline{(1-\varphi(z))(A(z, D)\tilde{G}(z, x)-g(z, x))}\rangle \\ &= \int_{\Omega_\varepsilon} A(z, D)G(\varepsilon, z, y)\varphi(z)\overline{\tilde{G}(z, x)}dz \\ &\quad + \int_{\Omega_\varepsilon} A(z, D)G(\varepsilon, z, y)(1-\varphi(z))\overline{\tilde{G}(z, x)}dz \\ &\quad + \int_{\Omega_\varepsilon} G(\varepsilon, z, y)\overline{g(z, x)}dz \\ &\quad + \sum_j \int_{\gamma_\varepsilon} B_j(\varepsilon, z, D)G(\varepsilon, z, y)\overline{S'_j(\varepsilon, z, D)\tilde{G}(z, x)}d\sigma^\varepsilon(z) \\ &\quad - \sum_j \int_{\gamma_\varepsilon} S_j(\varepsilon, z, D)G(\varepsilon, z, y)\overline{B'_j(\varepsilon, z, D)\tilde{G}(z, x)}d\sigma^\varepsilon(z). \end{aligned} \tag{3.3}$$

Here we used the Green formula (2.3). Since (2.9) holds, this is equal to

$$\begin{aligned} \langle \overline{\tilde{G}(z, x)}, \delta(z-y)\rangle &- \sum_j \int S_j(\varepsilon, z, D)G(\varepsilon, z, y)\overline{B'_j(\varepsilon, z, D)\tilde{G}(z, x)}d\sigma^\varepsilon(z) \\ &+ \int G(\varepsilon, z, y)\overline{g(z, x)}dz. \end{aligned}$$

Since $G(y, x)=\overline{G(x, y)}$, we obtain that

$$\begin{aligned} G(\varepsilon, x, y) &- G(x, y) \\ &= -\sum_j \int_{\gamma_\varepsilon} S_j(\varepsilon, z, D)G(\varepsilon, z, y)\overline{B'_j(\varepsilon, z, D)\tilde{G}(z, x)}d\sigma^\varepsilon(z) \\ &\quad + \int_{\Omega_\varepsilon} G(\varepsilon, z, y)\overline{g(z, x)}dz. \end{aligned} \tag{3.4}$$

As a consequence of (3.2), we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{\Omega_\varepsilon} G(\varepsilon, z, y)\overline{g(z, y)}dz=0. \tag{3.5}$$

It follows from (2.4) that

$$\begin{aligned} B'_j(\varepsilon, z, D)\tilde{G}(z, x) &= \sum_{k=1}^m T_j^k(\varepsilon, z, D)B_k(\varepsilon, z, D)\tilde{G}(z, x) \\ &\quad + \sum_{k=1}^m T_j^{k+m}(\varepsilon, z, D)S_k(\varepsilon, z, D)\tilde{G}(z, x). \end{aligned} \tag{3.6}$$

Applying (2.7), we have

$$T_j^{k+m}(\varepsilon, z, D)S_k(\varepsilon, z, D)\tilde{G}(z, x)=\varepsilon H_j^{k+m}(\varepsilon, z, D)S_k(\varepsilon, z, D)\tilde{G}(z, x). \tag{3.7}$$

On the other hand,

$$B_k(\varepsilon, z, D)\tilde{G}(z, x) = \varepsilon \frac{\partial}{\partial \varepsilon}(B_k(\varepsilon, z, D)\tilde{G}(z, x))\Big|_{\substack{z=w+\varepsilon\rho(w)\\ w\in\gamma}} + 0(\varepsilon^2), \tag{3.8}$$

because $B_k(0, z, D)G(z, x)|_{z=w\in\gamma}=0$ for any w in γ. Therefore,

$$\begin{aligned} &B_k(\varepsilon, z, D)\tilde{G}(z, x) \\ &= \varepsilon\Big(\frac{\partial}{\partial \varepsilon}B_k(0, z, D)\Big)G(z, x)\Big|_{z\in\gamma} + \varepsilon\rho(z)\frac{\partial}{\partial \nu_z}(B_k(z, D)G(z, x))\Big|_{z\in\gamma} + 0(\varepsilon^2). \end{aligned} \tag{3.9}$$

Dividing both sides of (3.4) by ε and using (3.5), (3.6) and (3.9), we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{-1}(G(\varepsilon, x, y) - G(x, y)) \\ &\quad = \lim_{\varepsilon\to 0} \sum_j \int_{\gamma_\varepsilon} \overline{S_j(\varepsilon, z, D)G(\varepsilon, z, D)} \\ &\quad\quad \times \Big\{ \overline{\frac{\partial}{\partial\varepsilon}B_j(0, z, D)\tilde{G}(z, x) + \rho(z)\frac{\partial}{\partial\nu_z}(B_j(z, D)G(z, x))} \\ &\quad\quad + \sum_k \overline{H_j^{k+m}(\varepsilon, z, D)S_k(\varepsilon, z, D)\tilde{G}(z, x)}\Big\} d\sigma^\varepsilon(z). \end{aligned} \tag{3.10}$$

The limit in the right hand side of (3.10) exists, because a priori estimates for the coercive elliptic boundary value problem (2.8) hold uniformly with respect to the parameter ε. The variational formula (2.10) is an immediate consequence of (3.10). More detailed proof will be published elsewhere.

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