55. *Asymptotic Behavior of Iterates of Nonexpansive Mappings in Banach Spaces*

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1. Introduction. Let X be a real Banach space and let X^* be the dual space of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The *duality mapping* F (multi-valued) from X into X^* is defined by

$$F(x)=\{x^* \in X^* : (x, x^*)=\|x\|^2 \text{ and } \|x^*\|=\|x\|\} \qquad \text{for } x \in X.$$

We say that X is *smooth*, if $\lim_{t\to 0} t^{-1}(\|x+ty\|-\|x\|)$ exists for every x and y with $\|x\|=\|y\|=1$ (i.e., the norm of X is Gâteaux differentiable). It is shown that F is single-valued if and only if X is smooth. The duality mapping F of a smooth Banach space X is said to be *weakly continuous* at 0 if $w\text{-}\lim_{n\to\infty} x_n=0$ in X implies that $\{F(x_n)\}$ converges weakly* to 0 in X^*, where $w\text{-}\lim_{n\to\infty} x_n$ denotes the weak limit of $\{x_n\}$. It is easily seen that Hilbert space and (l^p), $1<p<\infty$, have this property.

Throughout the rest of this paper we assume that X is a smooth and uniformly convex real Banach space having the duality mapping F which is weakly continuous at 0, and C is a nonempty closed convex subset of X. A mapping $T: C\to C$ is said to be *nonexpansive* on C, or $T \in \mathrm{Cont}(C)$ if $\|Tx-Ty\|\leqq\|x-y\|$ for all $x, y \in C$. The set of fixed-points of T will be denoted by $\mathcal{F}(T)$, i.e., $\mathcal{F}(T)=\{x \in C : Tx=x\}$.

The purpose of this note is to prove the following

Theorem. *Let $T \in \mathrm{Cont}(C)$ and $x \in C$. The following three conditions are mutually equivalent:*

(i) *$w\text{-}\lim_{n\to\infty} T^n x$ exists;*

(ii) *$\mathcal{F}(T)\neq\phi$ and $\omega_w(x)\subset\mathcal{F}(T)$;*

(iii) *$E(x)\neq\phi$ and $\omega_w(x)\subset E(x)$;*

where $\omega_w(x)$ denotes the set of weak subsequential limits of $\{T^n x\}$, and $E(x)=\{u \in C : \|T^n x-u\| \text{ converges as } n\to\infty\}$. Moreover, if $w\text{-}\lim_{n\to\infty} T^n x$ exists, then it is the asymptotic center of $\{T^n x\}$ with respect to C.

In Hilbert space, the equivalence of (i) and (ii) in Theorem has been established by A. Pazy [5]. As corollaries of Theorem, we have the following:

Corollary 1 (Z. Opial [4]). *Let $T \in \mathrm{Cont}(C)$ and $x \in C$. If $\mathcal{F}(T)\neq\phi$ and $\|T^{n+1}x-T^n x\|\to 0$ as $n\to\infty$, then the sequence $\{T^n x\}$ is weakly convergent to an element of $\mathcal{F}(T)$.*

Corollary 2 (R. E. Bruck [2]). *Let X be a real Hilbert space, and let $T \in \mathrm{Cont}(C)$ and $x \in C$. Then $w\text{-}\lim_{n\to\infty} T^n x$ exists if and only if $\mathcal{F}(T) \neq \phi$ and $w\text{-}\lim_{n\to\infty}(T^{n+1}x - T^n x) = 0$. Moreover, $w\text{-}\lim_{n\to\infty} T^n x \in \mathcal{F}(T)$ (if the limit exists).*

2. Proofs. Let $\{x_n\}$ be a bounded sequence in C and set

$$r(x) = \limsup_{n\to\infty} \|x_n - x\| \qquad \text{for } x \in X.$$

M. Edelstein [3] proved that there is a unique point $c \in C$ such that

$$r(c) < r(x) \qquad \text{for } x \in C \setminus \{c\}.$$

The point c is called the *asymptotic center* of $\{x_n\}$ with respect to C. We start with the following

Lemma. *Let $\{x_n\}$ be a bounded sequence in C, and let c be the asymptotic center of $\{x_n\}$ with respect to C.*

(a) *Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and let $w\text{-}\lim_{k\to\infty} x_{n_k} = u$. If $\{\|x_n - u\|\}$ is convergent, then $u = c$.*

(b) *If $w\text{-}\lim_{n\to\infty} x_n$ exists, then $w\text{-}\lim_{n\to\infty} x_n = c$.*

Proof. Note that

$$(2.1) \qquad \|x_n - c\|^2 \geqq \|x_n - z\|^2 + 2(z - c, F(x_n - z)) \qquad \text{for } z \in X.$$

To prove (a), let $w\text{-}\lim_{k\to\infty} x_{n_k} = u$ and $\{\|x_n - u\|\}$ be convergent. Letting $n \to \infty$ in (2.1) with z replaced by u, we have

$$r(c)^2 \geqq \lim_{n\to\infty} \|x_n - u\|^2 + 2 \limsup_{n\to\infty} (u - c, F(x_n - u)).$$

Since F is weakly continuous at 0,

$$\limsup_{n\to\infty} (u - c, F(x_n - u)) \geqq \lim_{k\to\infty} (u - c, F(x_{n_k} - u)) = 0.$$

Thus we have that $r(c)^2 \geqq \lim_{n\to\infty} \|x_n - u\|^2$, i.e., $r(c) \geqq r(u)$. Noting $u \in C$, it follows from the definition of c that $u = c$. To prove (b), let $w\text{-}\lim_{n\to\infty} x_n = v$. Letting $n \to \infty$ in (2.1) with z replaced by v, we have

$$r(c)^2 \geqq \limsup_{n\to\infty} \|x_n - v\|^2 + 2 \lim_{n\to\infty} (v - c, F(x_n - v)) = r(v)^2$$

by the weak continuity of F at 0. This implies that $v = c$. Q.E.D.

Proof of Theorem. Put $x_n = T^n x$ for $n = 1, 2, \cdots$. Suppose that $w\text{-}\lim_{n\to\infty} x_n$ exists. Then $\{x_n\}$ is bouned in C, and hence the asymptotic center c of $\{x_n\}$ with respect to C exists. It follows from Lemma (b) that $w\text{-}\lim_{n\to\infty} x_n = c$. Since c is a fixed-point of T (see [3, Theorem 1]), $\omega_w(x) = \{c\} \subset \mathcal{F}(T)$. Thus (i) implies (ii). Noting that $\{\|T^n x - u\|\}$ is monotone nonincreasing for every $u \in \mathcal{F}(T)$, we have that $\mathcal{F}(T) \subset E(x)$ and hence (ii) implies (iii). To show that (iii) implies (i), let $E(x) \neq \phi$ and $\omega_w(x) \subset E(x)$. By $E(x) \neq \phi$, $\{x_n\}$ is a bounded sequence in C and hence $\omega_w(x) \neq \phi$. Let $u \in \omega_w(x)$. By $\omega_w(x) \subset E(x)$, $\{\|x_n - u\|\}$ is convergent. It follows from Lemma (a) that $u = c$, where c is the asymptotic center of $\{x_n\}$ with respect to C. This means that $\omega_w(x) = \{c\}$, i.e., $w\text{-}\lim_{n\to\infty} x_n = c$. Q.E.D.

Proof of Corollary 1. By virtue of Theorem, it suffices to show that $\omega_w(x) \subset \mathcal{F}(T)$. Let $y \in \omega_w(x)$. There exists a subsequence $\{n_k\}$ of $\{n\}$ such that $w\text{-}\lim_{k\to\infty} T^{n_k} x = y$. Moreover, $\|(I - T)T^{n_k} x\| \to 0$ as $k \to \infty$ by our assumption. Thus $y \in \mathcal{F}(T)$. (For example, see [1, Proposition

1.3].) Q.E.D.

Proof of Corollary 2. By virtue of Theorem, it suffices to show that if $\mathcal{F}(T)\neq\phi$ and $w\text{-}\lim_{n\to\infty}(T^{n+1}x-T^nx)=0$ then $\omega_w(x)\subset E(x)$. To prove this, let $u\in\omega_w(x)$ and $w\text{-}\lim_{j\to\infty}T^{k_j}x=u$. Now, we want to show that $\{\|T^nx-u\|\}$ is convergent. To this end, take an $f\in\mathcal{F}(T)$ and set $x_n=T^nx-f$. By $w\text{-}\lim_{n\to\infty}(x_{n+1}-x_n)=w\text{-}\lim_{n\to\infty}(T^{n+1}x-T^nx)=0$, we have

$$w\text{-}\lim_{k\to\infty}(x_{k+m}-x_k)=0 \qquad \text{for every } m=1,2,\cdots. \tag{2.2}$$

Since

$$(x_m-x_n,x_k)=[(x_m,x_{k+m})-(x_n,x_{k+n})]+[(x_n,x_{k+n}-x_k)-(x_m,x_{k+m}-x_k)],$$

it follows from (2.2) that

$$\begin{aligned}(x_m-x_n,u-f)&(=\lim_{j\to\infty}(x_m-x_n,x_{k_j}))\\ &\leqq\limsup_{k\to\infty}(x_m-x_n,x_k)\\ &\leqq\limsup_{k\to\infty}[(x_m,x_{k+m})-(x_n,x_{k+n})].\end{aligned}\tag{2.3}$$

Note that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\limsup_{k\to\infty}[(x_m,x_{k+m})-(x_n,x_{k+n})]\leqq 0$$

(see [2, Proof of Theorem 1.1]). Combining this with (2.3) we have that

$$\limsup_{m\to\infty}(x_m,u-f)\leqq\liminf_{n\to\infty}(x_n,u-f),$$

i.e., $\{(x_n,u-f)\}$ is convergent. Moreover $\{\|x_n\|\}$ is convergent. Therefore $\|T^nx-u\|^2=\|x_n\|^2-2(x_n,u-f)+\|u-f\|^2$ is also convergent.

Q.E.D.

Added in Proof. After this paper was submitted for publication the author obtained the following which is an extension of Corollary 2: Let X, C, T and x be as in Theorem. Then $w\text{-}\lim_{n\to\infty}T^nx$ exsists if and only if $\mathcal{F}(T)\neq\phi$ and $w\text{-}\lim_{n\to\infty}(T^{n+1}x-T^nx)=0$.

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