53. *Reparametrization and Equicontinuous Flows*

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Let X be a topological space, and R denotes the set of real numbers. A continuous mapping $\pi: X\times R \to X$ is said to be *a dynamical system or a flow on (a phase space)* X if π satisfies the following two conditions:

(1) $\pi(x,0)=x$ for $x\in X$,

(2) $\pi(\pi(x,t),s)=\pi(x,t+s)$ for $x\in X$ and $t,s\in R$.

$C_\pi(x)$ denotes the orbit of π through $x\in X$. In this paper we always assume that phase spaces of flows are compact and connected metric spaces, and that every flow admits no singular point ($x\in X$ is called *a singular point of* π if $C_\pi(x)=\{x\}$). A flow π is said to be *equicontinuous* if $\{\pi_t\}_{t\in R}$ forms an equicontinuous family of homeomorphism of X onto Y, where π_t is defined by $\pi_t(x)=\pi(x,t)$ for $x\in X$. Let π and ρ be flows on X and Y, respectively. A homeomorphism h of X onto Y is called *an isomorphism of* π *onto* ρ if $h(C_\pi(x))=C_\rho(h(x))$ for $x\in X$. In this case, it is known ([1]) that there exists a continuous function $\phi: X\times R\to R$, which is called the reparametrization for h, satisfying $h(\pi(x,t))=\rho(h(x),\phi(x,t))$ for $(x,t)\in X\times R$. We can easily verify the above reparametrization ϕ satisfies the following condition (A):

(A) $\phi(x,t+s)=\phi(\pi(x,t),s)+\phi(x,t)$ for $x\in X$ and $t,s\in R$.

Further, if the both flows are equicontinuous, then ϕ is uniformly continuous on $X\times R$ ([2]). In this paper we shall show the following

Theorem. *Let π be an equicontinuous flow on X, and let ϕ be a continuous function on $X\times R$ satisfying the property* (A). *If ϕ is uniformly continuous on $X\times R$, then there exist a real number α and a continuous function $\Phi: X\to R$ satisfying*

$$\phi(x,t)=-\Phi(\pi(x,t))+\Phi(x)+\alpha t \qquad \textit{for } (x,t)\in X\times R.$$

To prove the theorem, we need several lemmas. Put $F_t(x)=\dfrac{\phi(x,t)}{t}$ for $(x,t)\in X\times[1,\infty)$.

Lemma 1. *$\{F_t\}_{t\geq 1}$ is uniformly bounded and equicontinuous.*

Proof. Equicontinuity of $\{F_t\}$ follows from the uniform continuity of ϕ. By the property (A) we have

$$\phi(x,t)=\phi(\pi(x,t-1),1)+\phi(x,t-1)$$

$$\phi(x,t)=\sum_{k=1}^{[t]}\phi(\pi(x,t-k),1)+\phi(x,t-[t])$$

for $(x,t)\in X\times[1,\infty)$. It follows that

$$\begin{aligned}|F_t(x)|&\leqq\frac{1}{t}([t]+1)M_1\\&=\frac{[t]}{t}\left(1+\frac{1}{[t]}\right)M_1\leqq 2M_1\end{aligned}$$

for $(x,t)\in X\times[1,\infty)$, where $M_1=\sup_{x\in X,|t|\leqq 1}\{|\phi(x,t)|\}$. Consequently, $\{F_t\}_{t\geqq 1}$ is uniformly bounded.

Lemma 2. *F_t converges uniformly to a constant as $t\to\infty$.*

Proof. At first, we shall show F_n (n: integer) converges as $n\to\infty$. Put $f(x)=\phi(x,1)$ and $H(x)=\pi(x,1)$ for $x\in X$, and f is continuous on X and H is a homeomorphism of X onto X. By equicontinuity of π, we can see that the powers $\{H^k\}_{k=1,2,\dots}$ of H forms an equicontinuous family of homeomorphisms of X onto X. Thus, since for each n and for $x\in X$

$$\begin{aligned}F_n(x)&=\frac{1}{n}\sum_{k=1}^{n}\phi(\pi(x,n-k),1)\\&=\frac{1}{n}\sum_{k=1}^{n}f(H^{n-k}(x))=\frac{1}{n}\sum_{k=0}^{n-1}f(H^k(x)),\end{aligned}$$

$\lim_{n\to\infty}F_n(x)$ exists for each $x\in X$ ([4]). Further, we have

$$\begin{aligned}|F_t(x)-F_{[t]}(x)|&=\left|\frac{\phi(x,t)}{t}-\frac{\phi(x,[t])}{[t]}\right|\\&=\left|\frac{\phi(\pi(x,[t]),t-[t])+\phi(x,[t])}{t}-\frac{\phi(x,[t])}{[t]}\right|\\&\leqq\left|\frac{\phi(\pi(x,[t]),t-[t])}{t}\right|+\left|\frac{\phi(x,[t])}{[t]}\left(\frac{[t]}{t}-1\right)\right|\\&\leqq\frac{M_1}{t}+F_{[t]}(x)\left(1-\frac{[t]}{t}\right)\to 0\end{aligned}$$

as $t\to\infty$. It follows that $\lim_{t\to\infty}F_t(x)$ exists for each $x\in X$, and hence, by Lemma 1, there exists a continuous function $\alpha: X\to R$ such that $F_t\to\alpha$ uniformly as $t\to\infty$.

Let $x_0\in X$ be fixed, and let $A=\{x\in X;\alpha(x)=\alpha(x_0)\}$. Then A is closed, because α is continuous. Further, A is open in X. In fact, by uniform continuity of ϕ, for each $x\in A$ there exists a $\delta>0$ such that $\sup_{t\in R}\{|\phi(x,t)-\phi(y,t)|\}\leqq 1$ for $y\in X$ with $d_X(x,y)<\delta$. For this y we have $|F_t(x)-F_t(y)|\leqq\frac{1}{t}$ for $t\geqq 1$, and hence, we have $\alpha(x)=\alpha(y)$, i.e., $y\in A$. This implies A is open in X. Since X is connected, we have $A=X$. Thus a continuous function α must be a constant.

Put $\psi(x, t)=\phi(x, t)-\alpha t$ for $(x, t) \in X\times R$, where α is the constant in Lemma 2.

Lemma 3. *ψ is uniformly continuous and bounded on $X\times[0, \infty)$.*

Proof. Uniform continuity of ψ follows from uniform continuity of ϕ. Let $x \in X$ be fixed, and choose a $\delta>0$ so that $|\phi(x, t)-\phi(y, t)|<1$ for $(y, t) \in X\times R$ with $d_X(x, y)<\delta$. Then we can show that $|\psi(x, t_0)| \leqq 1$ for $t_0 \in [0, \infty)$ satisfying $d_X(x, \pi(x, t_0))<\delta$. In fact, by the property (A), we have

$$(\,!\,)\qquad \psi(x, nt_0)=\psi(\pi(x, t_0), (n-1)t_0)+\psi(x, t_0)$$

$$\vdots$$

$$\psi(x, nt_0)=n\psi(x, t_0)+\sum_{k=1}^{n-1}\{\psi(\pi(x, t_0), kt_0)-\psi(x, kt_0)\}.$$

Put $\psi(\pi(x, t_0), kt_0)-\psi(x, kt_0)=\varepsilon_k$, and $|\varepsilon_k|<1$, because $\psi(\pi(x, t_0), kt_0)-\psi(x, kt_0)=\phi(\pi(x, t_0), kt_0)-\phi(x, kt_0)$. By (!) we obtain

$$\begin{aligned}\left|\frac{\psi(x, nt_0)}{nt_0}\right| &\geqq \left|\frac{\psi(x, t_0)}{t_0}\right|-\frac{|\varepsilon_1|+|\varepsilon_2|+\cdots+|\varepsilon_{n-1}|}{nt_0}\\ &\geqq \left|\frac{\psi(x, t_0)}{t_0}\right|-\frac{n-1}{nt_0}\\ &\geqq \frac{1}{t_0}(|\psi(x, t_0)|-1).\end{aligned}$$

Since the left side of the above inequality tends to 0 as $n\to\infty$ by Lemma 2, we have $|\psi(x, t_0)|\leqq 1$. Since π is equicontinuous, the closure $\overline{C_\pi(x)}$ of $C_\pi(x)$ is a minimal set of π ([3]). Thus there exists a relative dense subset $\{s_n\}\subset R$ such that $0<s_{n+1}-s_n\leqq L$ for some $L>0$ and $d_X(x, \pi(x, s_n))<\delta$ ([4]). By the proceeding assertion, we have $|\psi(x, s_n)| \leqq 1$ for each n. For each $t \in [0, \infty)$ we can find n such that $s_n\leqq t<s_{n+1}$ and we have

$$\begin{aligned}|\psi(x, t)|&=|\psi(x, s_n+(t-s_n))|\\ &=|\psi(\pi(x, s_n), t-s_n)+\psi(x, s_n)|\\ &\leqq|\phi(\pi(x, s_n), t-s_n)|+|\alpha|\,|t-s_n|+|\psi(x, s_n)|\\ &\leqq M_L+|\alpha|\,L+1,\end{aligned}$$

where $M_L=\sup_{x\in X,\ |t|<L}\{|\phi(x, t)|\}$. This implies that for each $x \in X$ there exists a $M_x>0$ such that $|\psi(x, t)|\leqq M_x$ for all $t\geqq 0$. Further, for each $x \in X$ there exists a $\delta_x>0$, by uniform continuity of ψ, such that $|\psi(x, t)-\psi(y, t)|<1$ for $t\geqq 0$ and $y \in X$ with $d_X(x, y)<\delta_x$. This implies, by compactness of X, that ψ is bounded on $X\times[0, \infty)$.

Proof of Theorem. Put

$$\Phi_t(x)=\frac{1}{t}\int_0^t \psi(x, s)ds \qquad (t\geqq 1,\ x \in X).$$

Then, by Lemma 3, $\{\Phi_t\}_{t\geqq 1}$ is equicontinuous and uniformly bounded. Hence, by Ascori-Alzera's theorem, there exists a sequences $\{c_n\}\subset R$ $(c_n\to\infty)$ and a continuous function $\Phi: X\to R$ such that $\Phi_{c_n}\to\Phi$ uniformly as $n\to\infty$. For each n and $t\in R$ we have

$$\begin{aligned}\Phi_{c_n}(\pi(x,t))&=\frac{1}{c_n}\int_0^{c_n}\psi(\pi(x,t),s)ds\\&=\frac{1}{c_n}\int_0^{c_n}\{\psi(x,t+s)-\psi(x,t)\}ds\\&=-\psi(x,t)+\frac{1}{c_n}\int_0^{c_n}\psi(x,t+s)ds\\&=-\psi(x,t)+\frac{1}{c_n}\int_0^{c_n}\psi(x,s)ds+\alpha_n,\end{aligned}$$

where $\alpha_n=\frac{1}{c_n}\int_{c_n}^{c_n+t}\psi(x,s)ds-\frac{1}{c_n}\int_0^t\psi(x,s)ds$. Since ψ is uniformly bounded on $X\times[0,\infty)$ by Lemma 3, we have $|\alpha_n|\to 0$ as $n\to\infty$. Thus we obtain

$$\begin{aligned}\Phi(\pi(x,t))&=-\psi(x,t)+\Phi(x)\\&=-\phi(x,t)+\alpha t+\Phi(x),\end{aligned}$$

because $\Phi_{c_n}\to\Phi$ uniformly as $n\to\infty$.

Remark 1. In the theorem, $\alpha=\lim_{t\to\infty}\frac{\phi(x,t)}{t}$. If π is minimal, then it is known ([4]) that π is strictly ergodic. Let μ be a unique invariant measure of π. In this case, if there exists a continuous function $H: X\to R$ such that $\phi(x,t)=\int_0^t H(\pi(x,s))ds$ for $(x,t)\in X\times R$, then we have $\alpha=\int_X H(x)d\mu(x)$.

Remark 2. In the theorem, $g_x(t)=\phi(x,t)-\alpha t$ is an almost periodic function for $x\in X$.

References

[1] M. Bebutov et V. V. Stepanov: Sur la measure invariante dans les systèmes dynamiques que ne diffèrent que par le temps. Mat. Sbornik, 143–166 (1949).
[2] J. Egawa: Distal minimal flows and equicontinuous minimal flows (in preparation).
[3] R. Ellis: Lecture on Topological Dynamics, W. A. Benjamin (1969).
[4] V. V. Nemytskii and V. V. Stepanov: Qualitative Theory of Differential Equations, Princeton Univ. Press, Princeton (1960).
[5] J. C. Oxtoby: Ergodic sets. Bull. Amer. Math. Soc., **58**, 116–136 (1952).