# 51. On Some Unilateral Problem of Elliptic and Parabolic Type

By Hiroki TANABE
Department of Mathematics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1978)

In this note we establish some regularity result for the parabolic unilateral problem

$$\partial u/\partial t + Lu \geqq f, \qquad u \geqq \Psi,$$
$$(\partial u/\partial t + Lu - f)(u-\Psi) = 0$$

as well as some related result for the associated elliptic problem.

Let $\Omega$ be a not necessarily bounded domain of $R^n$ with smooth boundary $\Gamma$. Let

$$a(u, v) = \int_\Omega \left( \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^N b_i \frac{\partial u}{\partial x_i} v + cuv \right) dx$$

be a bilinear form defined on the space $H^1(\Omega)\times H^1(\Omega)$ of real valued functions with real coefficients $a_{ij} \in B^1(\bar{\Omega})$, $a_i \in B^1(\bar{\Omega})$, $c \in L^\infty(\Omega)$, where $B^1(\bar{\Omega})$ is the set of functions continuous and bounded in $\bar{\Omega}$ together with first derivatives. Assume that the matrix $\{a_{ij}(x)\}$ is uniformly positive definite in $\Omega$ and there exists a positive number $\alpha$ such that

$$c \geqq \alpha, \; c - \sum_{i=1}^N \partial b_i / \partial x_i \geqq \alpha \quad \text{a.e.}$$

Let

$$L = -\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left( a_{ij} \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^N b_i \frac{\partial}{\partial x_i} + c$$

be the differential operator associated with the bilinear form $a(u, v)$. For $1 \leqq p \leqq \infty$ we denote by $L_p$ the realization of $L$ in $L^p(\Omega)$ under the Dirichlet boundary condition (refer to [2] or [6] for this subject where $\Omega$ is assumed to be bounded). Let $\Psi$ be a function defined in $\Omega$.

($\Psi$.1) For some $p$, $1<p<\infty$, $\Psi \in W^{2,p}(\Omega)$ and $\Psi|_\Gamma \leqq 0$.

($\Psi$.2) $\Psi \in W^{1,1}(\Omega)$, $L\Psi \in L^1(\Omega)$ and $\Psi|_\Gamma \leqq 0$.

By $M_p$ we denote the multivalued mapping defined by

$D(M_p) = \{u \in L^p(\Omega) : u \geqq \Psi \text{ a.e. in } \Omega\}$,

$M_p u = \{g \in L^p(\Omega) : g \leqq 0 \text{ a.e. in } \Omega,\ g(x) = 0 \text{ where } u(x) > \Psi(x)\}$.

When the assumption ($\Psi$.1) is satisfied, we define the operator $A_p$ by $A_p = L_p + M_p$; when the assumption ($\Psi$.2) as well as ($\Psi$.1) for some $1<p<\infty$ is satisfied, we define the operator $A_1$ by $A_1 = L_1 + M_1$.

**Proposition 1.** *$A_p$ and $A_1$ are m-accretive in $L^p(\Omega)$ and $L^1(\Omega)$ respectively and*

$$\overline{D(A_p)}=\{u \in L^p(\Omega): u \geqq \Psi \text{ a.e. in } \Omega\},$$
$$\overline{D(A_1)}=\{u \in L^1(\Omega): u \geqq \Psi \text{ a.e. in } \Omega\}.$$

By $G(A)$ we denote the graph of the mapping $A$.

**Theorem 1.** (i) *Suppose that* ($\Psi$.1) *is satisfied for some* $p$ *with* $1<p<\infty$. *Then, for any* $q$ *with* $p<q\leqq Np/(N-2p)$ *the operator* $A_q$ *defined by*

$$G(A_q)=\text{the closure of } G(A_p)\cap(L^q(\Omega)\times L^q(\Omega)) \text{ in } L^q(\Omega)\times L^q(\Omega) \tag{1}$$

*is m-accretive in* $L^q(\Omega)$ *and*

$$\overline{D(A_q)}=\{u \in L^q(\Omega): u \geqq \Psi \text{ a.e. in } \Omega\}. \tag{2}$$

(ii) *Suppose that* ($\Psi$.2) *as well as* ($\Psi$.1) *for some* $p$ *with* $1<p<\infty$ *is satisfied. Then for any* $1<q<p$ *the operator* $A_q$ *defined by* (1) *is m-accretive in* $L^q(\Omega)$ *and* (2) *holds.*

**Outline of the proof.** If $f \in L^p(\Omega)\cap L^q(\Omega)$, then $u=(I+A_p)^{-1}f$ is the limit of the solution of the approximate equation

$$u_\lambda+L_p u_\lambda+(u_\lambda-Pu_\lambda)/\lambda=f,$$

where $P$ is the operator defined by $Pw=\max\{w,\Psi\}$. Since $f \in L^q(\Omega)$ this equation may be written as

$$u_\lambda+L_q u_\lambda+(u_\lambda-Pu_\lambda)/\lambda=f.$$

Similarly, if $\hat{f}$ is another element of $L^p(\Omega)\cap L^q(\Omega)$, $\hat{u}=(1+A_p)^{-1}\hat{f}$ is the limit of the solution of

$$\hat{u}_\lambda+L_q\hat{u}_\lambda+(\hat{u}_\lambda-P\hat{u}_\lambda)/\lambda=\hat{f}.$$

Since $L_q$ and $(I-P)/\lambda$ are both accretive in $L^q(\Omega)$ we get $\|u_\lambda-\hat{u}_\lambda\|_q \leqq \|f-\hat{f}\|_q$. Going to the limit we obtain $\|u-\hat{u}\|_q \leqq \|f-\hat{f}\|_q$ which plays the fundamental role in the proof of the theorem.

By Theorem 1 the $m$-accretive operator $A_q$ is defined and (2) holds for all $q$ with $1\leqq q\leqq Np/(N-2p)$ if the assumptions ($\Psi$.1) and ($\Psi$.2) are satisfied.

In what follows we assume that ($\Psi$.1) and ($\Psi$.2) are satisfied for some $p$ satisfying $1<p<2$ and $p^*=(p^{-1}-N^{-1})^{-1}\geqq 2$. In this case $2\leqq(N-2)p/(N-2p)^{-1}<Np/(N-2p)$, hence by Theorem 1 the operator $A_2$ is defined and $m$-accretive in $L^2(\Omega)$. Furthermore, by Sobolev's imbedding theorem $\Psi$ belongs to $H^1(\Omega)$.

Let $\phi$ be the functional on $L^2(\Omega)$ defined by

$$\phi(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega\sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx+\frac{\alpha}{2}\int_\Omega u^2dx & \text{if } \Psi\leqq u\in H_0^1(\Omega),\\ \infty & \text{otherwise,}\end{cases}$$

and $B=\sum_{i=1}^N b_i\partial/\partial x_i+c-\alpha$ be the differential operator defined on $H_0^1(\Omega)$.

**Proposition 2.** $A_2=\partial\phi+B$.

Next, we consider the semilinear parabolic equation

$$du(t)/dt+A_q u(t)\ni f(t), \qquad 0<t\leqq T, \tag{3}$$
$$u(0)=u_0. \tag{4}$$

According to [4] we consider the solution of (3)–(4) constructed by

$$u(t)=\lim_{n\to\infty}\prod_{i=1}^{n}\left\{I+\frac{t}{n}\left(A_q-f\left(\frac{i}{n}t\right)\right)\right\}^{-1}u_0. \qquad (5)$$

**Theorem 2.** *If for some* $q$ *with* $1\leqq q\leqq 2$ $\Psi\leqq u_0\in L^q(\Omega)$ *and* $f\in W^{1,1}(0,T;\ L^q(\Omega)\cap L^r(\Omega))$, *then the function constructed by* (5) *is differentiable in* $L^r(\Omega)$ *for any* $r\geqq 2$ *and satisfies the equation*

$$du(t)/dt+\partial\phi(u(t))+Bu(t)\ni f(t) \text{ a.e. in } (0,T).$$

*There exists a constant* $C$ *depending on* $q$ *and* $r$ *such that*

$$\begin{aligned}&\|du(t)/dt\|_r\\ &\leqq C(1+\sqrt{t})t^{\beta-1}\Big\{\|\Psi\|_2+\|v\|_2+(t\phi(v))^{1/2}+t\,\|Bv\|_2\\ &\quad+t^{\gamma}\,\|u_0\|_q+t^{1-\delta}\,\|(L\Psi)^+\|_p+\int_0^t\|f(s)\|_2\,ds\Big\}\\ &\quad+Ct^{\beta}\int_0^t\|df(s)/ds\|_2\,ds+\int_0^t\|df(s)/ds\|_r\,ds\end{aligned}$$

*for any* $v\in D(\phi)$ *where* $\beta=N(r^{-1}-2^{-1})/2$, $\gamma=N(2^{-1}-q^{-1})/2$, $\delta=N(p^{-1}-2^{-1})/2$ *and* $\|\ \ \|_r$ *denotes the norm of* $L^r(\Omega)$.

Similar results remain valid for more general boundary condition

$$-\partial u/\partial\nu(x)\in\beta(x,u(x)) \qquad \text{on } \Gamma\times(0,T),$$

where $\beta(x,r)$ is maximal monotone in $R\times R$ for any $x\in\Gamma$.

In the proof of the results stated above essential use is made of the methods of [1], [3], and [6].

## References

[1] H. Brezis: Problèmes unilateraux. J. Math. Pure Appl., **51**, 1–168 (1972).

[2] H. Brezis and W. A. Strauss: Semi-linear second-order elliptic equations in $L^1$. J. Math. Soc. Japan, **25**, 565–590 (1973).

[3] B. D. Calvert and C. P. Gupta: Nonlinear elliptic boundary value problems in $L^p$-spaces and sums of ranges of accretive operators. Nonlinear Anal., **2**, 1–26 (1978).

[4] M. G. Crandall and A. Pazy: Nonlinear evolution equations in Banach spaces. Israel J. Math., **11**, 57–94 (1972).

[5] L. C. Evans: Regularity properties for the heat equation subject to nonlinear boundary constraints. Nonlinear Anal., **1**, 593–602 (1977).

[6] F. J. Massey, III: Semilinear parabolic equations with $L^1$ initial data. Indiana Univ. Math. J., **26**, 399–412 (1977).