50. *Meromorphic Functions on Compact Riemann Surfaces*

By Makoto NAMBA*)
Tohoku University

(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1978)

1. By a complex space, we mean a reduced, Hausdorff, complex analytic space. Let V be a compact Riemann surface of genus g. The set $\mathrm{Hol}(V, \boldsymbol{P}^1)$ of all holomorphic maps of V into the complex projective line $\boldsymbol{P}^1$ is nothing but the set of all meromorphic functions on V. A general theorem of Douady [1] says that $\mathrm{Hol}(V, \boldsymbol{P}^1)$ is a complex space. $\mathrm{Hol}(V, \boldsymbol{P}^1)$ is divided into the open (and closed) subspaces:

$$\mathrm{Hol}(V, \boldsymbol{P}^1) = \mathrm{Const} \cup R_1(V) \cup R_2(V) \cup \cdots,$$

where Const is the set of all constant functions and $R_n(V)$ is the set of all meromorphic functions on V of (mapping) order n. Note that $R_n(V)$ is non-empty for $n \geqq g+1$. Moreover, if $n \geqq g$, then $R_n(V)$ is non-singular and of dimension $2n+1-g$ (see [3, Proposition 5]). The automorphism group $\mathrm{Aut}(\boldsymbol{P}^1)$ of $\boldsymbol{P}^1$ acts freely and properly on $R_n(V)$ (see [3]). Hence the quotient space $R_n(V)/\mathrm{Aut}(\boldsymbol{P}^1)$ is a complex space and the projection $R_n(V) \to R_n(V)/\mathrm{Aut}(\boldsymbol{P}^1)$ is a principal $\mathrm{Aut}(\boldsymbol{P}^1)$-bundle (see Holmann [2]).

It is a difficult problem to determine the integers $n \leqq g$ with non-empty $R_n(V)$ and to determine the structure of $R_n(V)$ for such n. In this note, we state the following theorems. Details will be published elsewhere.

Theorem 1. *Let $V = C$ be a non-singular plane curve of degree $d \geqq 2$. Then*

$$\mathrm{Min}\{n > 0 \mid R_n(C) \textit{ is non-empty}\} = d-1.$$

If $d \geqq 3$, then $R_{d-1}(C)/\mathrm{Aut}(\boldsymbol{P}^1)$ is biholomorphic to C.

Theorem 2. *Let V be a compact Riemann surface of genus g. Let m and n be positive integers such that* (1) *m and n are relatively prime,* (2) $(m-1)(n-1) \leqq g-1$. *Then, at least one of $R_m(V)$ and $R_n(V)$ is empty.*

Corollary. *Let V be a compact Riemann surface of genus g. Let p be a prime number such that $R_p(V)$ is non-empty and let n be a positive integer such that $(p-1)(n-1) \leqq g-1$. Then,*

$$R_n(V) \begin{cases} \textit{is empty, if } n \not\equiv 0 \pmod{p} \\ \cong R_{n/p}(\boldsymbol{P}^1), \textit{ if } n \equiv 0 \pmod{p}. \end{cases}$$

*) Supported by Alexander von-Humboldt Foundation.

2. For $g \geqq 2$, let T_g be the Teichmüller space of compact Riemann surfaces of genus g. For a point $t \in T_g$, let V_t be the compact Riemann surface corresponding to t. For $n \geqq 2$, we put

$$R_n = \bigcup_{t \in T_g} R_n(V_t) \qquad \text{(disjoint union)}.$$

Theorem 3. *R_n is a non-singular complex space of dimension $2n+2g-2$.*

Again, $\mathrm{Aut}(\boldsymbol{P}^1)$ acts freely and properly on R_n. Hence

Corollary. *$R_n/\mathrm{Aut}(\boldsymbol{P}^1)$ is a non-singular complex space of dimension $2n+2g-5$.*

Now, we put

$$T_g(n) = \{t \in T_g \mid R_n(V_t) \text{ is non-empty}\}.$$

Applying the corollaries of Theorems 2 and 3, we can prove

Theorem 4. *Let p be a prime number such that $(p-1)^2 \leqq g-1$. Then*

(1) *$T_g(p)$ is an open subspace of a closed complex subspace of T_g and is of dimension $2p+2g-5$.*

(2) *$T_g(p)$ is singular at $t \in T_g(p)$ if and only if* $\dim |2D_\infty(f)| > 2$, *for $f \in R_p(V_t)$. ($D_\infty(f)$ is the polar divisor of f.)*

Corollary. (1) (Rauch [4]) *If $g \geqq 2$, then $T_g(2)$, the hyperelliptic locus, is a non-singular closed complex subspace of T_g of dimension $2g-1$.*

(2) *If $g \geqq 5$, then $T_g(3)$, the locus of trigonal compact Riemann surfaces, is non-singular and of dimension $2g+1$.*

(3) *If $p \geqq 5$ is a prime number such that $(p-1)(2p-3) \leqq g-1$, then $T_g(p)$ is non-singular.*

References

[1] Douady, A.: Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. Ann. Inst. Fourier, **16**, 1–95 (1966).

[2] Holmann, H.: Quotienten komplexer Räume. Math. Ann., **142**, 407–440 (1961).

[3] Namba, M.: Moduli of open holomorphic maps of compact complex manifolds. Ibid., **220**, 65–76 (1976).

[4] Rauch, H. E.: Weierstrass points, branch points and the moduli of Riemann surfaces. Comm. Pure. Appl. Math., **12**, 543–560 (1959).