46. A Remark on Bounded Reinhardt Domains

By Akio KODAMA*)
Department of Mathematics, Akita University

(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1978)

Introduction. Let D be a domain in N-complex Euclidian space C^N. We denote by Aut (D) the group of all biholomorphic automorphisms of D. In this note we prove the following

Theorem. *Let D be a bounded Reinhardt domain in C^N. Suppose that there exists a compact subset K of D such that* $\mathrm{Aut}(D)\cdot K=D$. *Then D is holomorphically equivalent to a finite product of unit open balls $B_i\subset C^{n_i}$ $(1\leqq i\leqq r)$: $D\cong B_1\times\cdots\times B_r$.*

Our proof is based on a recent work on bounded Reinhardt domains in C^N due to Sunada [3].

In the theory of bounded domains in C^N there is an outstanding conjecture as follows (cf. [2, p. 128]): If D is a bounded domain in C^N and if there exists a discrete subgroup Γ of Aut (D) such that D/Γ is compact, then D is homogeneous. In Vey [5], it was shown that this conjecture is true in the case when D is a generalized Siegel domain in $C^n\times C^m$ in the sense of Kaup, Matsushima and Ochiai [1]. So far as the author knows, this seems to be the only known result concerning this conjecture. Our result shows that, in the special case in which D is a bounded Reinhardt domain in C^N, not only the conjecture is true but the structure of D is completely determined.

The author wishes to express his thanks to Professor S. Murakami for his kind advices.

1. Sunada's results. Let D be a bounded Reinhardt domain in C^N. Then, by Sunada [3] there exist a coordinate system $(z^1,\cdots,z^N)$ in C^N and a bounded Reinhardt domain $\tilde{D}$ in C^N with center o, the origin of C^N, which is holomorphically equivalent to D and is described as follows (see [3] for the precise notations). For $(z^1,\cdots,z^N)$, we put

$$z_i=(z^{n_1+\cdots+n_{i-1}+1},\cdots,z^{n_1+\cdots+n_i})\qquad \text{for } 1\leqq i\leqq r,$$
$$w_j=(z^{s+m_1+\cdots+m_{j-1}+1},\cdots,z^{s+m_1+\cdots+m_j})\qquad \text{for } 1\leqq j\leqq t,$$
$$|z_i|^2=|z^{n_1+\cdots+n_{i-1}+1}|^2+\cdots+|z^{n_1+\cdots+n_i}|^2,$$

where $s=n_1+\cdots+n_r$ and $s+m_1+\cdots+m_t=N$. Then we have

Theorem A (Sunada [3]). (i) *Denoting by* $\mathrm{Aut}_0(\tilde{D})$ *the identity component of the Lie group* $\mathrm{Aut}(\tilde{D})$, *we put* $D_0=\mathrm{Aut}_0(\tilde{D})\cdot o$. *Then we have*

$$D_0=\{(z_1,\cdots,z_r,w_1,\cdots,w_t)\in C^N\,|\,|z_1|<1,\cdots,|z_r|<1,w_1=\cdots=w_t=0\}.$$

*) Supported in part by Yukawa Foundation.

(ii) $D_1=\{(w_1, \cdots, w_t)\in C^{N-s} \mid (0, \cdots, 0, w_1, \cdots, w_t)\in \tilde{D}\}$ *is a bounded Reinhardt domain in* C^{N-s}.

(iii) $\tilde{D}=\Big\{(z_1, \cdots, z_r, w_1, \cdots, w_t)\in C^N \mid (z_1, \cdots, z_r)\in D_0,$

$$\left(\frac{w_1}{(1-|z_1|^2)^{p_1^1/2}\cdots(1-|z_r|^2)^{p_1^r/2}}, \cdots, \frac{w_t}{(1-|z_1|^2)^{p_t^1/2}\cdots(1-|z_r|^2)^{p_t^r/2}}\right)\in D_1\Big\}.$$

Theorem B (Sunada [3]). *The group* $\mathrm{Aut}_0(\tilde{D})$ *consists of transformations of the following type*:

$$\begin{cases} z_i \mapsto (A^i z_i + b^i)\cdot(c^i z_i + d^i)^{-1}, \\ w_k \mapsto B^k \cdot \prod_{i=1}^{r} (c^i z_i + d^i)^{-p_k^i}\cdot w_k, \end{cases}$$

where $A^i\in \mathrm{Mat}(n_i\times n_i)$, $b^i \in \mathrm{Mat}(n_i\times 1)$, $c^i\in \mathrm{Mat}(1\times n_i)$, $d^i\in \mathrm{Mat}(1\times 1)$, $B^k\in U(m_k)$ *(unitary matrix), and they satisfy the following relations*:

$$(*)\qquad {}^t\bar{A}^i A^i - {}^t\bar{c}^i c^i = I_{n_i}, \quad {}^t\bar{b}^i b^i - |d^i|^2 = -1, \quad {}^t\bar{b}^i A^i - \bar{d}^i c^i = 0.$$

2. Proof of Theorem. We may assume that D is a bounded Reinhardt domain $\tilde{D}$ as in Theorem A. Under this assumption we show the following

Lemma. *Let D be a bounded Reinhardt domain in C^N. Suppose that there exists a compact subset K of D such that* $\mathrm{Aut}(D)\cdot K=D$. *Then there exists a compact subset $\tilde{K}$ of D such that* $\mathrm{Aut}_0(D)\cdot\tilde{K}=D$.

Proof. We may suppose that D is non-homogeneous. There exists a subset $S=\{g_\gamma \mid \gamma\in I\}$ of $\mathrm{Aut}(D)$ such that $\mathrm{Aut}(D)=\bigcup_{\gamma\in I}\mathrm{Aut}_0(D)\cdot g_\gamma$. Now, according to Sunada [4] we can find $g_{0\gamma}\in \mathrm{Aut}_0(D)$ and $l_\gamma\in L$ in such a way that $g_\gamma=g_{0\gamma}\cdot l_\gamma$ for each $g_\gamma\in S$, where L denotes the isotropy subgroup of $\mathrm{Aut}(D)$ at the origin o. Indeed, since D is non-homogeneous, the orbit $\mathrm{Aut}_0(D)\cdot o$ is of lowest dimension in the set of all $\mathrm{Aut}_0(D)$-orbits by Theorem B, i.e., $\dim(\mathrm{Aut}_0(D)\cdot o)<\dim(\mathrm{Aut}_0(D)\cdot(z, w))$ for any $(z, w)\in D-\mathrm{Aut}_0(D)\cdot o$. From this we see that $g_\gamma\cdot\mathrm{Aut}_0(D)\cdot o=\mathrm{Aut}_0(D)\cdot o$, which assures the existence of an element $g_{0\gamma}\in \mathrm{Aut}_0(D)$ such that $l_\gamma=g_{0\gamma}{}^{-1}\cdot g_\gamma\in L$, as is claimed. Now, since the isotropy subgroup L is compact, the set $\tilde{K}=L\cdot K$ is also compact in D. We see that this set $\tilde{K}$ is a required one in our lemma. Indeed, by our choice of the elements $g_{0\gamma}$ and l_γ we have

$$\begin{aligned} D=\mathrm{Aut}(D)\cdot K &= \bigcup_{\gamma\in I}\mathrm{Aut}_0(D)\cdot g_\gamma\cdot K=\bigcup_{\gamma\in I}\mathrm{Aut}_0(D)\cdot g_{0\gamma}\cdot l_\gamma\cdot K \\ &\subset \bigcup_{\gamma\in I}\mathrm{Aut}_0(D)\cdot\tilde{K}=\mathrm{Aut}_0(D)\cdot\tilde{K}\subset D, \end{aligned}$$

and so $\mathrm{Aut}_0(D)\cdot\tilde{K}=D$, completing the proof.

Proof of Theorem. Put $G=\mathrm{Aut}_0(D)$. By virtue of Theorem A it is enough to show that D is homogeneous. Suppose that D is non-homogeneous. Then, the w-part appears in Theorem A. By our lemma we may assume that $G\cdot K=D$ from the beginning. We define a mapping $F: D \to D_1$ by

$$(z_1, \cdots, z_r, w_1, \cdots, w_t) \mapsto \left(\frac{w_1}{(1-|z_1|^2)^{p_1^1/2} \cdots (1-|z_r|^2)^{p_1^r/2}}, \cdots, \frac{w_t}{(1-|z_1|^2)^{p_t^1/2} \cdots (1-|z_r|^2)^{p_t^r/2}} \right),$$

where D_1 is the domain defined in Theorem A. By Theorem A F is a well-defined continuous mapping. Thus the image $F(K)=K_1$ is also compact in D_1. Putting $U=U(m_1)\times \cdots \times U(m_t)$, we define an action of U on D_1 in a canonical manner and set $K_2=U\cdot K_1$. Then Theorem B assures that K_2 is a compact subset of D_1. From now on, we identify a subset A of D_1 with the subset (o, A) of D. Now, we claim that $G\cdot K_2=D$. Indeed, since $G\cdot K=D$, for any point $(z, w)\in K$ there exists an element $g\in G$ such that $g\cdot(z,w)=(o,\tilde{w})$ for some $\tilde{w}\in D_1$, where $(z,w)=(z_1,\cdots,z_r,w_1,\cdots,w_t)$ and $\tilde{w}=(\tilde{w}_1,\cdots,\tilde{w}_t)$. Let

$$(\sharp) \qquad g: \begin{cases} z_i \mapsto (A^i z_i + b^i)\cdot(c^i z_i + d^i)^{-1} \\ w_k \mapsto B^k \cdot \prod_{i=1}^{r} (c^i z_i + d^i)^{-p_k^i} \cdot w_k. \end{cases}$$

Then we have $(A^i z_i + b^i)\cdot(c^i z_i+d^i)^{-1}=o$. On the other hand, by using the relations $(*)$ in Theorem B we see that $|c^i z_i+d^i|^2-|A^i z_i+b^i|^2=1-|z_i|^2$. Thus it follows from these two equalities that $|c^i z_i+d^i|^2=1-|z_i|^2$. Putting $\theta_i=arg\cdot(c^i z_i+d^i)$, we have then

$$(c^i z_i+d^i)^{-p_k^i}=\exp(-\sqrt{-1}p_k^i\theta_i)\cdot(1-|z_i|^2)^{-p_k^i/2},$$

and hence

$$\tilde{w}_k=B^k\cdot\exp\left\{-\sqrt{-1}\left(\sum_{i=1}^{r} p_k^i\theta_i\right)\right\}\cdot\prod_{i=1}^{r}(1-|z_i|^2)^{-p_k^i/2}\cdot w_k.$$

It follows that $g\cdot(z,w)=(o, B\cdot F(z,w))$, where $B=B_1\times\cdots\times B_t$ and $B_k=B^k\cdot\exp\left\{-\sqrt{-1}\left(\sum_{i=1}^{r} p_k^i\theta_i\right)\right\}\in U(m_k)$. This implies that $B\cdot F(z,w)\in U\cdot K_1=K_2$, and so $K\subset G\cdot K_2$. Therefore, by our assumption we have $D=G\cdot K=G\cdot K_2$. Now, we assert that $G\cdot K_2\cap D_1=K_2$. Once this is shown, our proof is completed, because in this case we have a contradiction $D_1=K_2$. Now, it suffices to show that $K_2\supset G\cdot K_2\cap D_1$. Take a point $\tilde{w}$ of $G\cdot K_2\cap D_1$ arbitrarily. Then there exist $g\in G$ and $w\in K_2$ such that $g\cdot(o,w)=(o,\tilde{w})$. When g has the explicit form as in ($\sharp$), we see by a direct computation that $|d^i|=1$. This means that $B_k=B^k\cdot\prod_{i=1}^{r}(d^i)^{-p_k^i}$ also belongs to $U(m_k)$, and hence $B=B_1\times\cdots\times B_t\in U$. Since K_2 is U-invariant and $\tilde{w}=B\cdot w$, we have $\tilde{w}\in K_2$, completing the proof.

References

[1] W. Kaup, Y. Matsushima, and T. Ochiai: On the automorphisms and equivalences of generalized Siegel domains. Amer. J. Math., **92**, 475–498 (1970).

[2] R. Narasimhan: Several Complex Variables, The University of Chicago

Press (1971).
[3] T. Sunada: On Bounded Reinhardt Domains. Proc. Japan Acad., **50**, 119–123 (1974).
[4] ——: Holomorphic equivalence problem for bounded Reinhardt domains. Math. Ann., **235**, 111–128 (1978).
[5] J. Vey: Sur la division des domaines de Siegel. Ann. Sci. Éc. Norm. Sup. IV, **3**, 479–506 (1970).