43. On Linnik's Problem

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Introduction. Ju. V. Linnik (1950) posed the following problem. Let F be a finite extension of the rational number field $\boldsymbol{Q}$, K_i/F a finite extension, $i=1, \cdots, r$, $r \geqq 1$ an integer, χ_i a Grössencharacter of K_i, $L(s, \chi_i)=\sum_{\mathfrak{a}} c_i(\mathfrak{a})N(\mathfrak{a})^{-s}$ the Hecke L-function expanded over integral ideals $\mathfrak{a}$ of F with considering $L(s, \chi_i)$ as an Euler product over F in the natural way. Then is $L_F(s, \chi_1, \cdots, \chi_r)=\sum_{\mathfrak{a}} c_1(\mathfrak{a}) \cdots c_r(\mathfrak{a})N(\mathfrak{a})^{-s}$ meromorphic on the complex numbers $\boldsymbol{C}$? (Originally Linnik posed the problem for $F=\boldsymbol{Q}$.) The answer seemed to be affirmative in general and Draxl [1] proved the following theorem in a generalized form after some results by B. Z. Moroz and A. I. Vinogradov.

Theorem (Draxl). *If χ_i are unitary Grössencharacters* $(i=1, \cdots, r)$ *then $L_F(s, \chi_1, \cdots, \chi_r)$ is meromorphic in* $\mathrm{Re}(s)>0$.

The purpose of this paper is to show that $L_F(s, \chi_1, \cdots, \chi_r)$ is not necessarily meromorphic on $\boldsymbol{C}$. We prove following Theorem 1 using Theorem 2 of [3] whose proof is described in Part I of [4]. We follow the notations of [3].

Theorem 1. *Let $F/\boldsymbol{Q}$ be a finite extension, K_i/F a finite extension of degree $n_i=[K_i : F]$, χ_i a Grössencharacter of K_i of finite order, $i=1, \cdots, r$, $\boldsymbol{n}=(n_1, \cdots, n_r)$, $1 \leqq n_1 \leqq \cdots \leqq n_r$, $r \geqq 1$ an integer. Then:*

(1) *$\boldsymbol{n}$ is of type* $\mathrm{I} \Leftrightarrow L_F(s, \chi_1, \cdots, \chi_r)$ *is meromorphic on $\boldsymbol{C}$.*

(2) *$\boldsymbol{n}$ is of type* $\mathrm{II} \Leftrightarrow L_F(s, \chi_1, \cdots, \chi_r)$ *is meromorphic in* $\mathrm{Re}(s)>0$ *with the natural boundary* $\mathrm{Re}(s)=0$.

§ 1. Tensor products.

Let $F/\boldsymbol{Q}$ be a finite extension, K_i/F a finite extension, χ_i a Grössencharacter of K_i, $i=1, \cdots, r$, $r \geqq 1$ an integer. By considering $L(s, \chi_i)$ to be an Euler product over F in the natural way, we write $L(s, \chi_i)=\prod_{\mathfrak{p}} \det(1-M_{i,\mathfrak{p}}N(\mathfrak{p})^{-s})^{-1}$ where $\mathfrak{p}$ runs over all prime ideals of F and $M_{i,\mathfrak{p}}$ is a complex square matrix for each $\mathfrak{p}$. We put $L_F(s, \chi_1 \otimes \cdots \otimes \chi_r)=\prod_{\mathfrak{p}} \det(1-M_{1,\mathfrak{p}} \otimes \cdots \otimes M_{r,\mathfrak{p}}N(\mathfrak{p})^{-s})^{-1}$. Then this Euler product does not depend on the choice of $M_{i,\mathfrak{p}}$ and we call it the tensor product over F of $L(s, \chi_i)$, $i=1, \cdots, r$. $L_F(s, \chi_1, \cdots, \chi_r)$ (defined as in Introduction) is called the scalar product over F of $L(s, \chi_i)$, $i=1, \cdots, r$.

Theorem 2. *Let $F/\boldsymbol{Q}$, K_i/F, χ_i, $i=1, \cdots, r$ be as above. Then $L_F(s, \chi_1 \otimes \cdots \otimes \chi_r)$ is meromorphic on $\boldsymbol{C}$.*

Proof. Let $W_F=W(\bar{F}/F)$ and $W_i=W(\bar{F}/K_i)$ be the absolute Weil groups of F and K_i respectively, where $\bar{F}$ is an algebraic closure of F. We denote by W_i^{ab} the maximal abelian quotient of W_i as a topological group. We denote by C_i the idele class group of K_i. Since $W_i^{ab}\cong C_i$ and $\chi_i: C_i \to GL(1, \boldsymbol{C})$ is a continuous homomorphism, we can naturally regard χ_i as a continuous homomorphism $\chi_i: W_i \to GL(1, \boldsymbol{C})$. Let $\rho_i = \mathrm{Ind}\,(W_F, W_i; \chi_i)$ the induced representation. Since W_i is a subgroup of W_F of index $n_i=[K_i:F]$, we have a continuous homomorphism $\rho_i: W_F \to GL(n_i, \boldsymbol{C})$ and the following equality holds: $L(s,\chi_i)=L(s,\rho_i)$. Here $L(s,\rho_i)$ is the Artin-Hecke L-function defined by Weil [5]. Then it is obvious that $L_F(s,\chi_1\otimes\cdots\otimes\chi_r)=L(s,\rho_1\otimes\cdots\otimes\rho_r)$. (This and similar equations hereafter hold except for a finite number of Euler factors.) Hence $L_F(s,\chi_1\otimes\cdots\otimes\chi_r)$ is meromorphic on $\boldsymbol{C}$. Q.E.D.

Remark 1. Let $L(s, \boldsymbol{H}^i)$ $(i=1,\cdots,r;\ r\geqq 1$ an integer) be Euler products over F. Then the tensor product $L(s, \boldsymbol{H}^1\otimes\cdots\otimes \boldsymbol{H}^r)$ and the scalar product $L(s, \boldsymbol{H}^1,\cdots,\boldsymbol{H}^r)$ of these Euler products are defined in the same manner as above, and $L(s, \boldsymbol{H}^1\otimes\cdots\otimes \boldsymbol{H}^r)\cdot L(s, \boldsymbol{H}^1,\cdots,\boldsymbol{H}^r)^{-1}$ is an Euler product over F (cf. [4], Part I).

§2. Type I case. Theorem 3. *Let $F/\boldsymbol{Q}, K_i/F, \chi_i, i=1,\cdots,r$ be as in* §1. *Assume that $\boldsymbol{n}=(n_1,\cdots,n_r)$ with $n_i=[K_i:F]$ is of type* I. *Then $L_F(s,\chi_1,\cdots,\chi_r)$ is meromorphic on $\boldsymbol{C}$.*

Proof. An easy calculation shows that: $L_F(s,\chi_1,\cdots,\chi_r)=L_F(s,\chi_1\otimes\cdots\otimes\chi_r)$

$$\times\begin{cases}1 & \text{if } \boldsymbol{n}=(1,\cdots,1,*),\\ L(2s,\rho)^{-1} \text{ with } \rho=(\rho_1\cdots\rho_{r-2})^2\det(\rho_{r-1})\det(\rho_r) & \text{if } \boldsymbol{n}=(1,\cdots,1,2,2).\end{cases}$$

Here ρ_i is associated to χ_i as in the proof of Theorem 1. Hence $L_F(s,\chi_1,\cdots,\chi_r)$ is meromorphic on $\boldsymbol{C}$. Q.E.D.

Remark 2. The case $\boldsymbol{n}=(2,2)$ is treated in Fomenko [2] by a different method.

§3. Type II case. Theorem 4. *Let $F/\boldsymbol{Q}, K_i/F, \chi_i, i=1,\cdots,r$ be as in* §1. *Assume that $\boldsymbol{n}=(n_1,\cdots,n_r)$ with $n_i=[K_i:F]$ is of type* II *and χ_i are of finite order. Then $L_F(s,\chi_1,\cdots,\chi_r)$ is meromorphic in* $\mathrm{Re}\,(s)>0$ *with the natural boundary* $\mathrm{Re}\,(s)=0$.

Proof. To prove this we use Theorem 2 of [3] whose proof is described in [4] (Part I). Since $\chi_i: C_i\to GL(1,\boldsymbol{C})$ is a continuous homomorphism of finite order (C_i being as in §1), there exists a continuous homomorphism $\rho_i^0: \mathrm{Gal}\,(\bar{F}/K_i)\to GL(1,\boldsymbol{C})$ such that $L(s,\chi_i)=L(s,\rho_i^0)$ by Artin's reciprocity law, where $\bar{F}$ is an algebraic closure of F. Let $\rho_i=\mathrm{Ind}\,(\mathrm{Gal}\,(\bar{F}/F), \mathrm{Gal}\,(\bar{F}/K_i); \rho_i^0)$ the induced representation. Let $K=\bar{F}^N$ with $N=\mathrm{Ker}\,(\rho_1)\cap\cdots\cap\mathrm{Ker}\,(\rho_r)$, where $\bar{F}^N$ is the fixed field of N in $\bar{F}$. Since N is an open normal subgroup of $\mathrm{Gal}\,(\bar{F}/F)$, K/F is a finite Galois extension with $\mathrm{Gal}\,(K/F)\cong\mathrm{Gal}\,(\bar{F}/F)/N$. Then we can regard ρ_i as a homomorphism $\rho_i: \mathrm{Gal}\,(K/F)\to GL(n_i,\boldsymbol{C})$ in the natural way and $L(s, \rho_i^0) = L(s, \rho_i)$. Hence $L(s, \chi_i) = L(s, \rho_i)$ and $L_F(s, \chi_1, \cdots, \chi_r) = L(s, \rho_1, \cdots, \rho_r)$. To the right hand side we can apply Theorem 2 of [3] ((2)$\Rightarrow$) and we get Theorem 4. Q.E.D.

Theorem 1 stated in Introduction follows easily from Theorems 3 and 4. (Otherwise we may apply Theorem 2 of [3] directly.)

Remark 3. The condition that χ_i are of finite order is weakened to some extent by generalizing Theorem 2 of [3] to the case of representations of the Weil group $W(K/F)$. As a particular case of this generalization we have Draxl's theorem in Introduction also (cf. [4], Parts II and III).

References

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