42. *On the Meromorphy of Euler Products*

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Introduction. We extend ordinary L-functions slightly and study their meromorphy. For simplicity we describe here the results on Euler products of Artin type which are contained in Part I of [3]. In Parts II and III of [3] we have some generalizations and modifications. Detailed proofs are described in [3].

§ 1. Euler products of Artin type. Let F be a finite extension of the rational number field $\boldsymbol{Q}$, K/F a finite Galois extension with the Galois group $G=\mathrm{Gal}\,(K/F)$, $R(G)$ the character ring of G (i.e. the ring of virtual characters of G; representations are over the complex number field $\boldsymbol{C}$). For $g \in G$ (or for the conjugacy class of G containing g) and for $H(T) \in 1+T\cdot R(G)[T]$ where T is an indeterminate, we denote by $H_g(T) \in 1+T\cdot \boldsymbol{C}[T]$ the polynomial obtained from $H(T)$ by taking the values of the coefficients at g. For each prime ideal $\mathfrak{p}$ of F unramified in K/F, let $\alpha(\mathfrak{p})$ denote the Frobenius conjugacy class $\left[\frac{K/F}{\mathfrak{P}}\right]$ in G, where $\mathfrak{P}$ is a prime ideal of K dividing $\mathfrak{p}$. We define $L(s,H) = \prod_{\mathfrak{p}} H_{\alpha(\mathfrak{p})}(N(\mathfrak{p})^{-s})^{-1}$ where $\mathfrak{p}$ runs over all prime ideals of F unramified in K/F.

We say $H(T) \in 1+T\cdot\boldsymbol{C}[T]$ is unitary if there exists a (complex) unitary matrix M such that $H(T)=\det(1-MT)$. We say $H(T)=1$ is unitary. For an Euler product over F ($F/\boldsymbol{Q}$ being a finite extension) $L(s,\boldsymbol{H})=\prod_{\mathfrak{p}} H_{\mathfrak{p}}(N(\mathfrak{p})^{-s})^{-1}$ with $\boldsymbol{H}=(H_{\mathfrak{p}})_{\mathfrak{p}}$, $H_{\mathfrak{p}}(T) \in 1+T\cdot\boldsymbol{C}[T]$, where $\mathfrak{p}$ runs over all prime ideals of F, we say $L(s,\boldsymbol{H})$ is unitary if $H_{\mathfrak{p}}(T)$ are unitary for all $\mathfrak{p}$. In general if $H_{\mathfrak{p}}(T)$ is not defined for a prime ideal $\mathfrak{p}$ of F, then we consider $H_{\mathfrak{p}}(T)=1$. We remark that the unitariness of $L(s,\boldsymbol{H})$ is not altered when we consider $L(s,\boldsymbol{H})$ as an Euler product over $\boldsymbol{Q}$ in the natural way. More precisely if F_0 is a subfield of F, then we can consider $L(s,\boldsymbol{H})$ as an Euler product over F_0 in the natural way as follows: for each prime ideal $\mathfrak{q}$ of F_0, put $H_{\mathfrak{q}}(T)=\prod_{\mathfrak{p}|\mathfrak{q}} H_{\mathfrak{p}}(T^{f(\mathfrak{p}|\mathfrak{q})})$ where $\mathfrak{p}$ runs over all prime ideals of F dividing $\mathfrak{q}$ and $f(\mathfrak{p}|\mathfrak{q})$ is the relative degree of $\mathfrak{p}$ over $\mathfrak{q}$, then $L(s,\boldsymbol{H})=L(s,\boldsymbol{H}_0)$ with $\boldsymbol{H}_0=(H_{\mathfrak{q}})_{\mathfrak{q}}$. Under this process the unitariness is not altered. It may be remarked that the unitariness is an analogue of the (normalized) "Riemann-Ramanujan-Weil conjecture" or "temperedness" for some arithmetic objects.

Following Theorem 1 is a main result for Euler products of Artin type.

Theorem 1. *Let $F/\boldsymbol{Q}$, K/F, G, $H(T)\in 1+T\cdot R(G)[T]$, $L(s,H)$ be as above. Then*:

(1) *$L(s,H)$ is unitary$\Leftrightarrow$$L(s,H)$ is meromorphic on $\boldsymbol{C}$.*

(2) *$L(s,H)$ is not unitary$\Leftrightarrow$$L(s,H)$ is meromorphic in* $\mathrm{Re}(s)>0$ *with the natural boundary* $\mathrm{Re}(s)=0$; *each point on* $\mathrm{Re}(s)=0$ *is a limit-point of poles of $L(s,H)$ in* $\mathrm{Re}(s)>0$.

Remark 1. The case $F=K=\boldsymbol{Q}$ is treated in Estermann [2].

Example 1. Let $F/\boldsymbol{Q}$, K/F, $G=\mathrm{Gal}(K/F)$, $R(G)$ be as above. Let $\rho: G\to GL(n,\boldsymbol{C})$ be a homomorphism ($n\geqq 1$ being an integer), and put $P_\rho(T)=\det(1-\rho T)=\sum_{i=0}^{n}(-1)^i \operatorname{tr}(\wedge^i\rho)T^i\in 1+T\cdot R(G)[T]$ where T is an indeterminate and $\wedge^i\rho$ is the (equivalence class of) i-th exterior power of ρ. Then $L(s,P_\rho)$ is unitary, and it follows from Theorem 1 that $L(s,P_\rho)$ is meromorphic on $\boldsymbol{C}$. In fact $L(s,P_\rho)=L(s,\rho)$ is the Artin L-function (except for a finite number of Euler factors), and the meromorphy of $L(s,\rho)$ is due to R. Brauer. We use this fact in our proof of Theorem 1.

Example 2. Let the notations be as in Example 1. Let m be a non-zero integer. For each finite place v of F, let $t(v)=-\log(m)/\log(N(v))$ where $N(v)$ is the "norm" of v (i.e. the number of elements of the residue field at v) and $\log(m)$ is the principal value (but what we need is $N(v)^{-t(v)}=m$). Let $t(v)=0$ for each infinite place v of F. We denote by A_F the adele ring of F. For each idele $a=(a_v)_v\in GL(1,A_F)$, let $\omega_m(a)=\prod_v |a_v|_v^{t(v)}$ where $|\ |_v$ is the normalized valuation at v and v runs over all places of F. Then $\omega_m: GL(1,A_F)\to GL(1,\boldsymbol{C})$ is a continuous homomorphism and an admissible representation of $GL(1,A_F)$ in the usual sense. The corresponding L-function (the finite part) is $L(s,\omega_m)=\prod_{\mathfrak{p}}(1-m\cdot N(\mathfrak{p})^{-s})^{-1}$ where $\mathfrak{p}$ runs over all prime ideals of F. We define $L(s,\rho,\omega_m)=L(s,P_\rho^{(m)})$ where $P_\rho^{(m)}(T)=P_\rho(m\cdot T)\in 1+T\cdot R(G)[T]$. It follows from Theorem 1 that: (1) *If $|m|=1$, then $L(s,\rho,\omega_m)$ is meromorphic on $\boldsymbol{C}$.* (2) *If $|m|>1$, then $L(s,\rho,\omega_m)$ is meromorphic in* $\mathrm{Re}(s)>0$ *with the natural boundary* $\mathrm{Re}(s)=0$. In general if we assume the existence of admissible (and automorphic) representation $\pi(\rho)$ of $GL(n,A_F)$ attached to ρ in the usual sense, then essentially $L(s,\rho,\omega_m)=L(s,\pi(\rho)\otimes\omega_m)$. In particular if $n=1$, then $\pi(\rho)$ exists by Artin's reciprocity law and $\pi(\rho)\otimes\omega_m$ is an admissible (and not automorphic for $m\neq 1$) representation of $GL(1,A_F)$. More generally it follows from a generalization of Theorem 1 that if $\rho: W(K/F)\to GL(1,\boldsymbol{C})$ is a (continuous) one-dimensional unitary representation of the Weil group $W(K/F)$, then the above (1) and (2) hold for $L(s,\pi(\rho)\otimes\omega_m)$. This example is considered to be an example of the analytic behaviour of Euler products attached to admissible (not necessarily automorphic) representations.

§2. **Scalar products.** Let $F/\boldsymbol{Q}$, K/F, $G=\mathrm{Gal}(K/F)$ be as in §1. Let $\rho_i: G\to GL(n_i, \boldsymbol{C})$ be a homomorphism, $i=1, \cdots, r$, $r\geqq 1$. Let $L(s, \rho_i)=\sum_{\mathfrak{a}} c_i(\mathfrak{a})N(\mathfrak{a})^{-s}$ be the Artin L-function expanded over integral ideals $\mathfrak{a}$ of F. We call $L(s, \rho_1, \cdots, \rho_r)=\sum_{\mathfrak{a}} c_1(\mathfrak{a})\cdots c_r(\mathfrak{a})N(\mathfrak{a})^{-s}$ the scalar product of $L(s, \rho_i)$, $i=1, \cdots, r$.

For $\boldsymbol{n}=(n_1, \cdots, n_r)$ with $1\leqq n_1\leqq \cdots \leqq n_r$ integers, $r\geqq 1$, we make the following definition: $\boldsymbol{n}$ is of type I if $\boldsymbol{n}=(1, \cdots, 1, *)$ or $(1, \cdots 1, 2, 2)$ ($\boldsymbol{n}=(*), (1, *), (2,2)$ for $r\leqq 2$), and $\boldsymbol{n}$ is of type II if $\boldsymbol{n}$ is otherwise. Then we obtain following Theorem 2 from Theorem 1.

Theorem 2. *Let $F/\boldsymbol{Q}$, K/F, G, ρ_i, $i=1, \cdots, r$, $L(s, \rho_1, \cdots, \rho_r)$ be as above. Assume that $1\leqq n_1\leqq \cdots \leqq n_r$, and set $\boldsymbol{n}=(n_1, \cdots, n_r)$. Then:*

(1) *$\boldsymbol{n}$ is of type I $\Leftrightarrow L(s, \rho_1, \cdots, \rho_r)$ is meromorphic on $\boldsymbol{C}$.*

(2) *$\boldsymbol{n}$ is of type II $\Leftrightarrow L(s, \rho_1, \cdots, \rho_r)$ is meromorphic in* $\mathrm{Re}(s)>0$ *with the natural boundary* $\mathrm{Re}(s)=0$.

Remark 2. This result has an application to Linnik's problem, cf. [4].

§3. **An application.** We have an application of Theorem 1 to Dirichlet series attached to elliptic modular forms of weights one. We follow the notations of Deligne-Serre [1]. Let $f(z)=\sum_{n=0}^{\infty} a(n)q^n$ ($q=\exp(2\pi\sqrt{-1}\cdot z)$) be a holomorphic modular form on $\Gamma_0(N)$ ($N\geqq 1$ an integer) of type $(1, \varepsilon)$, ε being an odd character mod N. We assume that $f(z)$ is an eigen-function of Hecke operator $T(p)$ for each prime $p\nmid N$ and $f(z)$ is normalized ($a(1)=1$). For simplicity we say $f(z)$ is a holomorphic normalized eigen modular form. We have the following results from Theorem 1 (cf. Theorem 2) by applying the main result in Deligne-Serre [1].

Theorem 3. *Let $f_i(z)=\sum_{n=0}^{\infty} a_i(n)q^n$ be a holomorphic normalized eigen modular form on $\Gamma_0(N_i)$ of type $(1, \varepsilon_i)$, $i=1, \cdots, r$, $r\geqq 1$. Let $L(s, f_1, \cdots, f_r)=\sum_{n=1}^{\infty} a_1(n)\cdots a_r(n)n^{-s}$. Then:*

(1) *$r=1$ or $2 \Leftrightarrow L(s, f_1, \cdots, f_r)$ is meromorphic on $\boldsymbol{C}$.*

(2) *$r\geqq 3 \Leftrightarrow L(s, f_1, \cdots, f_r)$ is meromorphic in* $\mathrm{Re}(s)>0$ *with the natural boundary* $\mathrm{Re}(s)=0$.

Theorem 3–a. *Let $f(z)=\sum_{n=0}^{\infty} a(n)q^n$ be a holomorphic normalized eigen modular form on $\Gamma_0(N)$ of type $(1, \varepsilon)$. Let $m\geqq 1$ be an integer. Then:*

(1) *$m=1$ or $2 \Leftrightarrow \sum_{n=1}^{\infty} a(n)^m n^{-s}$ and $\sum_{n=1}^{\infty} a(n^m)n^{-s}$ are meromorphic on $\boldsymbol{C}$.*

(2) *$m\geqq 3 \Leftrightarrow \sum_{n=1}^{\infty} a(n)^m n^{-s}$ and $\sum_{n=1}^{\infty} a(n^m)n^{-s}$ are meromorphic in* $\mathrm{Re}(s)>0$ *with their natural boundaries* $\mathrm{Re}(s)=0$.

Remark 3. Similar results hold for holomorphic elliptic eigen cusp forms of weights $\geqq 2$ assuming a modification of Sato-Tate conjecture.

Remark 4. From a modification of Theorem 1, we have an application to the meromorphy of Dirichlet series constructed from the eigen-values of Hecke operators on Siegel modular forms of degree 3.

References

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