41. *On the Logarithmic Kodaira Dimension of the Complement of a Curve in* $\mathbf{P}^2$

By Isao WAKABAYASHI
Department of Mathematics, Tokyo University of Agriculture and Technology, Fuchu, Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., June 15, 1978)

1. The logarithmic Kodaira dimension introduced by S. Iitaka [1] plays an important role in the study of non-compact algebraic varieties. In this note we calculate the logarithmic Kodaira dimension $\bar{\kappa}(\boldsymbol{P}^2-C)$ of the complement of an irreducible curve C in the complex projective space $\boldsymbol{P}^2$ of dimension 2. We denote by $g(C)$ the genus of the non-singular model of C. In this note, a locally irreducible singular point of C will be called cusp. Our results are as follows:

Theorem. *Let C be an irreducible curve of degree n in $\boldsymbol{P}^2$.*

(I) *If $g(C)\geqslant 1$ and $n\geqslant 4$, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

(II) *If $g(C)=0$ and C has at least three cusps, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

(III) *If $g(C)=0$, C has at least two singular points, and one of the singular points is locally reducible, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

(IV) *If $g(C)=0$ and C has two cusps, then $\bar{\kappa}(\boldsymbol{P}^2-C)\geqslant 0$.*

For the definition of logarithmic Kodaira dimension, see S. Iitaka [1].

Remark 1. It is with ease to show that $\bar{\kappa}(\boldsymbol{P}^2-C)=0$ for any non-singular elliptic curve C of degree 3 in $\boldsymbol{P}^2$.

Remark 2. F. Sakai [5] and S. Iitaka [3], independently of us, showed the same result as Case (I).

2. **Monoidal transformations.** Let

$$\tilde{\boldsymbol{P}}^2=S_t\xrightarrow{\pi_t}S_{t-1}\longrightarrow\cdots\longrightarrow S_1\xrightarrow{\pi_1}\boldsymbol{P}^2$$

be a finite sequence of monoidal transformations with successive centers $p_1,\cdots,p_t$. We pose $\pi=\pi_1\circ\cdots\circ\pi_t:\tilde{\boldsymbol{P}}^2\to\boldsymbol{P}^2$. Let E_i be the exceptional curve of the monoidal transformation π_i. Let us denote by E_i' the proper transform of E_i by $\pi_{i+1}\circ\cdots\circ\pi_t$. By definition, E_i is a divisor in S_i, but we shall use for the sake of simplicity the same letter E_i for $(\pi_{i+1}\circ\cdots\circ\pi_t)^*E_i$ also. Let H be an arbitrary line in $\boldsymbol{P}^2$. We shall use the same letter H for π^*H also.

We frequently use the following lemma to calculate $\bar{\kappa}$.

Lemma. *Let $\pi:\tilde{\boldsymbol{P}}^2\to\boldsymbol{P}^2$, H and E_i be as above. Then we have for any $N\in\boldsymbol{N}$, $n_i\in\boldsymbol{N}\cup\{0\}$ the following:*

$$\dim H^0\Big(\tilde{\boldsymbol{P}}^2,\mathcal{O}\Big(NH-\sum_{i=1}^t n_iE_i\Big)\Big)\geqslant\frac{1}{2}(N+1)(N+2)-\sum_{i=1}^t\frac{1}{2}n_i(n_i+1).$$

Proof. It is sufficient to show the lemma for the case where the infinite line does not contain any p_i, and where H is the infinite line. A polynomial of degree N, $h=\sum_{\lambda+\mu\leqslant N} a_{\lambda\mu}x^\lambda y^\mu$, has multiplicity at least n_1 at $p_1=(x_1, y_1)$ if and only if its coefficients $a_{\lambda\mu}$ satisfy $n_1(n_1+1)/2$ linear equations:

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}h(x_1, y_1)=0, \qquad \text{for } \alpha, \beta \geqslant 0,\ \alpha+\beta \leqslant n_1-1.$$

Suppose that p_2 lies in E_1. Let $x-x_1=x'$, $y-y_1=x'y'$ be the monoidal transformation at p_1, and let us pose $h(x, y)=h(x_1+x', y_1+x'y')=x'^{n_1}h_2(x', y')$. Then each coefficient $a'_{\lambda\mu}$ of h_2 is a linear form of $\{a_{\lambda\mu}\}$. Consequently $h_2(x', y')$ has multiplicity at least n_2 at p_2 if and only if the coefficients $a_{\lambda\mu}$ satisfy $n_2(n_2+1)/2$ linear equations. We continue this process and we have $\sum_{i=1}^t n_i(n_i+1)/2$ linear equations of $a_{\lambda\mu}$. A polynomial h whose coefficients $a_{\lambda\mu}$ satisfy all these linear equations is exactly an element of $H^0(\tilde{\boldsymbol{P}}^2, \mathcal{O}(NH-\sum_{i=1}^t n_iE_i))$. Then the proof of the lemma follows quickly.

3. **Proof of Theorem.** Let C be an irreducible curve of degree $n\geqslant 4$ in $\boldsymbol{P}^2$. We perform a succession of monoidal transformations as in the preceding paragraph, and we use the same notations. We suppose that the (reduced) inverse image $\bar{D}=\pi^{-1}(C)$ is a divisor with normal crossings. Let m_i be the multiplicity at p_i of the proper transform of C by $\pi_{i-1}\circ\cdots\circ\pi_1$. Let C' be the proper transform of C by π. We denote by $\bar{K}$ the canonical bundle of $\tilde{\boldsymbol{P}}^2$. Then we have

$$\bar{D}=C'+\sum_{i=1}^t E'_i,$$

$$\bar{K}\sim -3H+\sum_{i=1}^t E_i, \qquad \text{(linearly equivalent)}$$

$$nH\sim C=C'+\sum_{i=1}^t m_iE_i,$$

and this implies

$$\bar{D}+\bar{K}\sim(n-3)H+\sum_{i=1}^t E'_i-\sum_{i=1}^t (m_i-1)E_i. \tag{1}$$

The assertions (I), (II), and (III) of the theorem will be derived from the following proposition:

Proposition 1. *Let $C, n, \bar{D}, \bar{K}$, and H be as above. Suppose we have the following relation for sufficiently large $k\in\boldsymbol{N}$:*

$$\alpha k(\bar{D}+\bar{K})\sim\alpha(n-3)H+\bar{D}_k \tag{2}$$

where α is a suitable positive number independent of k and $\bar{D}_k$ is a suitable non-negative divisor in $\tilde{\boldsymbol{P}}^2$ dependent on k. Then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.

Proof. Take an integer k such that (2) holds. Then for any $m\in\boldsymbol{N}$ we have

$$\dim H^0(\tilde{\boldsymbol{P}}^2, \mathcal{O}(m\alpha k(\bar{D}+\bar{K})))\geqslant\dim H^0(\tilde{\boldsymbol{P}}^2, \mathcal{O}(m\alpha(n-3)H)).$$

It is obvious that there exists a positive constant c independent of m

such that $\dim H^0(\tilde{\boldsymbol{P}}^2, \mathcal{O}(m\alpha(n-3)H))\geqslant cm^2$. By definition of the logarithmic Kodaira dimension, Proposition 1 follows immediately.

The relation (1) contains a negative divisor $-\sum_{i=1}^t (m_i-1)E_i$. This is inconvenient to calculate $\dim H^0(\tilde{\boldsymbol{P}}^2, \mathcal{O}(k(\bar{D}+\bar{K})))$. So we eliminate this negative part, as will be described in the following, by rewriting $(n-3)H$ through the usage of the above lemma so that we can obtain the equation (2) and derive the theorem from Proposition 1.

Case (I). *Suppose that* $g(C)\geqslant 1$ *and* $n\geqslant 4$. We apply the lemma to the case where $N=n-3$ and $n_i=m_i-1$ $(i=1,\cdots,t)$. Then the classical formula ([4], p. 393)

$$g(C)=\frac{1}{2}(n-1)(n-2)-\sum_{i=1}^t \frac{1}{2}m_i(m_i-1) \tag{3}$$

and the assumption $g(C)\geqslant 1$ show that

$$H^0\Big(\tilde{\boldsymbol{P}}^2, \mathcal{O}\Big((n-3)H-\sum_{i=1}^t (m_i-1)E_i\Big)\Big)\neq 0.$$

This asserts that $(n-3)H\sim\sum_{i=1}^t (m_i-1)E_i+C_1$, where C_1 is a positive divisor in $\tilde{\boldsymbol{P}}^2$. By this relation and (1), we have

$$\begin{aligned} k(\bar{D}+\bar{K})&\sim(n-3)H+(k-1)(n-3)H+k\sum_{i=1}^t E'_i-k\sum_{i=1}^t (m_i-1)E_i \\ &\sim(n-3)H+(k-1)C_1+k\sum_{i=1}^t E'_i-\sum_{i=1}^t (m_i-1)E_i. \end{aligned}$$

As E_i is a linear combination of E'_j $(j=1,\cdots,t)$, we obtain from this equation the desired relation (2) for sufficiently large k and for $\alpha=1$. So the assertion of (I) of the theorem follows from Proposition 1.

Case (II). Suppose, for the moment, that C is a curve of genus 0 with only one singular point p_1 and that it is a cusp. Let us denote by s the index such that the proper transform of the curve is singular at p_s and non-singular at p_{s+1} in the process of monoidal transformations. Let us further suppose that the number t of our monoidal transformations is the smallest one to obtain $\bar{D}$ with normal crossings. Then, by observing the diagram of monoidal transformations, we have

$$E_s=E'_s+E_{s+1}+\cdots+E_t, \qquad E_{t-1}=E'_{t-1}+E'_t, \tag{4}$$

$$t-s=m_s. \tag{5}$$

We apply the lemma to the following set:

$$H^0\Big(\tilde{\boldsymbol{P}}^2, \mathcal{O}\Big((n-3)H-\sum_{i\neq s} (m_i-1)E_i-(m_s-2)E_s-E_{s+1}-\cdots-E_{t-2}\Big)\Big).$$

By (3) and (5) we have

$$\frac{1}{2}(n-1)(n-2)-\frac{1}{2}\sum_{i\neq s} m_i(m_i-1)-\frac{1}{2}(m_s-1)(m_s-2)-\underbrace{1-\cdots-1}_{m_s-2}=1,$$

so the lemma shows that this set is not empty, and we have

$$\begin{aligned} (n-3)H\sim&\sum_{i\neq s} (m_i-1)E_i \\ &+(m_s-2)E_s+E_{s+1}+\cdots+E_{t-2}+C_2 \end{aligned} \tag{6}$$

where C_2 is a positive divisor.

Then we obtain the following relation from (1) and (6):

$$k(\bar{D}+\bar{K})\sim(n-3)H+(k-1)(n-3)H+k\sum_{i=1}^{t}E'_i-k\sum_{i=1}^{t}(m_i-1)E_i$$
$$\sim(n-3)H+(k-1)C_2-\sum_{i=1}^{t}(m_i-1)E_i+k\sum_{i=1}^{t}E'_i$$
$$+(k-1)(-E_s+E_{s+1}+\cdots+E_{t-2}).$$

So we have from (4)

$$k(\bar{D}+\bar{K})\sim(n-3)H+(k-1)C_2-\sum_{i=1}^{t}(m_i-1)E_i+k\sum_{i=1}^{t}E'_i \qquad (7)$$
$$-(k-1)(E'_s+E'_{t-1}+2E'_t).$$

Now *let us suppose that C has at least three cusps p_1, p_2, and p_3.* As above, let us denote by s_j $(j=1,2,3)$ the index such that the proper transform of the curve is singular at p_{s_j} and non-singular at p_{s_j+1} in the process of desingularization of the singular point p_j. We pose $t_j=s_j+m_{s_j}$. Then we obtain three equations analogous to (7) corresponding to $j=1,2,3$:

$$k(\bar{D}+\bar{K})\sim(n-3)H+(k-1)C_{2,j}-\sum_{i=1}^{t}(m_i-1)E_i+k\sum_{i=1}^{t}E'_i \qquad (7)'$$
$$-(k-1)(E'_{s_j}+E'_{t_j-1}+2E'_{t_j})$$

where t is the number of all monoidal transformations. By adding these three equations, we have

$$3k(\bar{D}+\bar{K})\sim3(n-3)H+(k-1)\sum_{j=1}^{3}C_{2,j}-3\sum_{i=1}^{t}(m_i-1)E_i+3k\sum_{i=1}^{t}E'_i$$
$$-(k-1)\sum_{j=1}^{3}(E'_{s_j}+E'_{t_j-1}+2E'_{t_j}).$$

As the indices $s_1, s_2, s_3, t_1-1, \cdots, t_3$ are all different, we obtain from this the desired relation (2) for large k and for $\alpha=3$.

Case (III). We use the following proposition in this case:

Proposition 2 (S. Iitaka [2] (Appendix)). *Let S be a non-singular compact projective surface such that $H^1(S,\mathcal{O}_s)=0$. Let $D=\sum_{i=1}^{r}C_i$ be a divisor in S with normal crossings, and with its irreducible components C_i. We denote by K the canonical bundle of S. Then we have*

$$\dim H^0(S,\mathcal{O}(D+K))=\operatorname{rank}H_1(D,\boldsymbol{Z})-\sum_{i=1}^{r}g(C_i)+\dim H^2(S,\mathcal{O}_S).$$

The proof of (III) will be divided in two cases (i) and (ii).

(i) *Suppose C has at least a cusp p_1 and a locally reducible singular point p_2, and is of genus* 0. Let us denote by I_j $(j=1,2)$ the set of all indices i such that the point p_i appears in the process of desingularization of the singular point p_j. We apply the above proposition to our surface $\tilde{\boldsymbol{P}}^2$ and the divisor $\bar{D}_{p_2}=C'+\sum_{i\in I_2}E'_i$. This is possible because $H^1(\tilde{\boldsymbol{P}}^2,\mathcal{O}_{\tilde{P}^2})=0$. As p_2 is a locally reducible singular point, we see easily that $\operatorname{rank}H_1(\bar{D}_{p_2},\boldsymbol{Z})\neq0$, and this implies that $H^0(\tilde{\boldsymbol{P}}^2,\mathcal{O}(\bar{D}_{p_2}+\bar{K}))\neq0$. So we have $\bar{D}_{p_2}+\bar{K}\sim C_3$ where C_3 is a nonnegative divisor, and this is equivalent to

$$(n-3)H \sim -\sum_{i\in I_2} E'_i + \sum_{i=1}^{t} (m_i-1)E_i + C_3. \qquad (8)$$

We have from (1) and (8)

$$k(\bar{D}+\bar{K}) \sim (n-3)H + (k-1)C_3$$
$$-\sum_{i=1}^{t} (m_i-1)E_i + k\sum_{i=1}^{t} E'_i - (k-1)\sum_{i\in I_2} E'_i. \qquad (9)$$

The equation (7)′ for $j=1$ and (9) imply

$$3k(\bar{D}+\bar{K}) \sim 3(n-3)H + (k-1)C_{2,1} + 2(k-1)C_3$$
$$-3\sum_{i=1}^{t} (m_i-1)E_i + 3k\sum_{i=1}^{t} E'_i - (k-1)(E'_{s_1} + E'_{t_1-1} + 2E'_{t_1})$$
$$-2(k-1)\sum_{i\in I_2} E'_i.$$

As the indices s_1, t_1-1 and t_1 are not contained in I_2, we obtain from this the desired relation (2) for large k.

(ii) *Suppose C has at least two locally reducible singular points p_1 and p_2, and is of genus* 0. Then we obtain, for p_1 also, an equation analogous to (9), and by adding this and (9) we have

$$2k(\bar{D}+\bar{K}) \sim 2(n-3)H + (k-1)C_3 + (k-1)C_4$$
$$-2\sum_{i=1}^{t} (m_i-1)E_i + 2k\sum_{i=1}^{t} E'_i - (k-1)\sum_{i\in I_1} E'_i - (k-1)\sum_{i\in I_2} E'_i$$

where C_4 is a non-negative divisor. Consequently we obtain (2) for large k.

Case (IV). *Suppose that C has two cusps p_1 and p_2, and is of genus* 0. Then we have two equations analogous to (6) corresponding to $j=1,2$:

$$(n-3)H \sim \sum_{i\neq s_j} (m_i-1)E_i + (m_{s_j}-2)E_{s_j} + E_{s_j+1} + \cdots + E_{t_j-2} + C_{2,j}.$$

From these two equations and (1), we have

$$2(\bar{D}+\bar{K}) \sim 2\sum_{i=1}^{t} E'_i + \sum_{j=1}^{2} C_{2,j} + \sum_{j=1}^{2} (-E_{s_j} + E_{s_j+1} + \cdots + E_{t_j-2}),$$

and further, from two equations analogous to (4), we have

$$2(\bar{D}+\bar{K}) \sim 2\sum_{i=1}^{t} E'_i + \sum_{j=1}^{2} C_{2,j} - \sum_{j=1}^{2} (E'_{s_j} + E'_{t_j-1} + 2E'_{t_j}).$$

The right hand side of this equation is a positive divisor, so

$$H^0(\tilde{P}^2, \mathcal{O}(2(\bar{D}+\bar{K}))) \neq 0.$$

Consequently we have, by definition, $\bar{\kappa}(\tilde{P}^2 - C) \geqslant 0$.

Remark. Let us suppose that we have, in the process of monoidal transformations, the same condition for the singularities of C as in Cases (II), (III), or (IV). Our demonstration is valid in this case also, and we have the same conclusion.

References

[1] S. Iitaka: On logarithmic Kodaira dimension of algebraic varieties. Complex Analysis and Algebraic Geometry (A collection of papers dedicated to K. Kodaira), Iwanami, pp. 175–189 (1977).

[2] S. Iitaka: On the Diophantine equation $\varphi(X,Y)=\varphi(x,y)$ (to appear in J. Reine Angew. Math.).
[3] ——: On logarithmic $K3$ surfaces (to appear).
[4] R. Hartshorne: Algebraic Geometry. Springer-Verlag (1977).
[5] F. Sakai: Logarithmic pluricanonical maps of algebraic surfaces (to appear).