40. The Intersection of Topologies

By Ken-ichi TAMANO
Tomioka High School, Kanagawa Prefecture

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Let X be a set, $\mathfrak{T}$ a topology on X, and $A \subset X$. We denote by $\overline{A}_{\mathfrak{T}}$ the closure of A in the space $(X, \mathfrak{T})$. Given two topologies $\mathfrak{T}_1, \mathfrak{T}_2$ on X, we say that $\mathfrak{T}_1, \mathfrak{T}_2$ are *compatible* if, for every $A \subset X$, $\overline{A}_{\mathfrak{T}_1 \cap \mathfrak{T}_2} = \overline{A}_{\mathfrak{T}_1} \cup \overline{A}_{\mathfrak{T}_2}$.

It is known by A. V. Arhangel'skiĭ [1, Theorem 2] that the intersection of two compatible topologies with uniform base (or point-countable base) is again a topology with uniform base (or point-countable base). A. V. Arhangel'skiĭ [1] raised the following questions:

(1) Is the intersection of two (Hausdorff, regular, completely regular) compatible topologies with development again a topology with development?

(2) Is the intersection of two (Hausdorff, regular, completely regular) compatible topologies with σ-disjoint base again a topology with σ-disjoint base?

In this paper, we shall give negative answers for these questions by showing counterexamples.

R denotes the set of real numbers, and N denotes the set of natural numbers.

Let

$$X = \{(x, y) \in R \times R : y \geqq 0\},$$
$$X_0 = \{(x, y) \in R \times R : y = 0\}.$$

The underlying set of each example is always the half upper plane X, and the points of $X - X_0$ are always isolated. So we shall only give a neighborhood base at each point of X_0.

1. Two developable compatible topologies whose intersection is not developable. $(X, \mathfrak{T}_1)$: For each $x_0 \in R$, and $n \in N$, let $U_n(x_0) = \{(x_0, 0)\} \cup \left\{(x, y) \in X : x_0 - \frac{1}{n} < x < x_0 + \frac{1}{n},\ 0 < y \leqq |x - x_0|\right\}$, and $\{U_n(x_0)\}_{n \in N}$ be a neighborhood base at $p = (x_0, 0) \in X_0$. Then the constructed space $(X, \mathfrak{T}_1)$ is a developable T_2-space.

$(X, \mathfrak{T}_2)$: Let $V_n(x_0) = \left\{(x, y) \in X : x_0 - \frac{1}{n} < x < x_0 + \frac{1}{n},\ y = 0\right\}$, and $\{V_n(x_0)\}_{n \in N}$ be a neighborhood base at $p = (x_0, 0) \in X_0$. Then $(X, \mathfrak{T}_2)$ is a metrizable space.

$(X, \mathfrak{T}_1 \cap \mathfrak{T}_2)$: Because $U_n(x_0) \cup V_n(x_0)$ is a neighborhood at $p = (x_0, 0)$

in the topology $\mathfrak{T}_1 \cap \mathfrak{T}_2$, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are compatible. The space $(X, \mathfrak{T}_1 \cap \mathfrak{T}_2)$ is a M_1-space, but is not metrizable because it contains a separable subspace which is not second countable, as described in E. van Douwen [2]. So $(X, \mathfrak{T}_1 \cap \mathfrak{T}_2)$ is not developable.

Remark. It is not known whether we can construct regular or completely regular counterexamples for this question or not.

2. Two compatible topologies with σ-disjoint base whose intersection is not a topology with σ-disjoint base. $(X, \mathfrak{T}_3)$: Let $U_n(x_0) = \left\{(x, y) \in X : y < \frac{1}{n}, y = x - x_0\right\}$, and $\{U_n(x_0)\}_{n \in N}$ be a neighborhood base at $p = (x_0, 0) \in X_0$. Then $(X, \mathfrak{T}_3)$ is a metrizable space.

$(X, \mathfrak{T}_4)$: Let $V_n(x_0) = \left\{(x, y) \in X : y < \frac{1}{n}, y = -x + x_0\right\}$, and $\{V_n(x_0)\}_{n \in N}$ be a neighborhood base at $p = (x_0, 0) \in X_0$. Then $(X, \mathfrak{T}_4)$ is a metrizable space.

$(X, \mathfrak{T}_3 \cap \mathfrak{T}_4)$: Topologies $\mathfrak{T}_3$ and $\mathfrak{T}_4$ are compatible, and the space $(X, \mathfrak{T}_3 \cap \mathfrak{T}_4)$ is a completely regular metacompact developable space which is not screenable as described in R. Heath [3, Example 1]. We can see that $(X, \mathfrak{T}_3 \cap \mathfrak{T}_4)$ has not a σ-disjoint base.

References

[1] A. V. Arhangel'skiĭ: The intersection of topologies and pseudo-open compact mappings. Dokl. Akad. Nauk SSSR, **226**(4) (1976) (in Russian); Soviet Math. Dokl., **17**(1), 160–163 (1976) (English translation).

[2] Eric K. van Douwen: Nonstratifiable regular quotients of separable stratifiable spaces. Proc. Amer. Math. Soc., **52**, 457–460 (1975).

[3] R. W. Heath: Screenability, pointwise paracompactness, and metrization of Moore spaces. Canad. J. Math., **16**, 763–770 (1964).