38. On L^2-Boundedness and L^2-Compactness of Pseudo-Differential Operators

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1. Introduction. After the works of Calderón-Vaillancourt [1], [2], Cordes [3] developed a new method by which, among others, the L^2-boundedness can be proved easily. In particular, he showed that if a symbol $a(x,\xi)$ defined on $R^n \times R^n$ has bounded derivatives $D_x^\alpha D_\xi^\beta a$ for $|\alpha|, |\beta| \leqq [n/2]+1$, then the pseudo-differential operator $A=a(X,D)$ is L^2-bounded. Subsequently Kato [4] formulated the basic idea of Cordes in a slightly different form and showed that the same method can be used to prove that A is L^2-bounded if $(1+|\xi|)^{(|\beta|-|\alpha|)\rho} D_x^\alpha D_\xi^\beta a$ is bounded for $|\alpha| \leqq [n/2]+2$, $|\beta| \leqq [n/2]+1$, where ρ is a constant such that $0<\rho<1$. These two results [3], [4] improve the order of the differentiability required in [1], [2] for simple symbols $a(x,\xi)$.

Applying the Sobolev lemma, a symbol satisfying one of the sufficient conditions above is necessarily continuous. In this paper, we obtain some classes of bounded pseudo-differential operators whose symbols are not necessarily continuous (see Theorems 2 and 3). The key point is the fact that if the derivatives of a symbol up to some order can be estimated by the L^p-norm $(1 \leqq p \leqq \infty)$, then the associated pseudo-differential operator is L^2-bounded (see Lemma 1). As corollaries, we obtain also the sufficient conditions for L^2-compactness.

2. Definitions and notations. Given any tempered distribution a on $R^n \times R^n$, the pseudo-differential operator $A=a(X,D)$ is defined by

$$(2.1)\qquad {}_{\mathcal{S}'(R^n)}\langle Au, v\rangle_{\mathcal{S}(R^n)} = {}_{\mathcal{S}'(R^n\times R^n)}\langle a, w\rangle_{\mathcal{S}(R^n\times R^n)},\quad w(x,\xi)=(2\pi)^{-n/2}e^{i\xi x}\hat{u}(\xi)v(x),$$

where $u, v \in \mathcal{S}(R^n)$ (the Schwartz space). As usual, (2.1) may be written symbolically as

$$(2.2)\qquad Au(x)=a(X,D)u(x)=(2\pi)^{-n/2}\int d\xi e^{i\xi x}a(x,\xi)\hat{u}(\xi).$$

If in particular $a(x,\xi)=x_j$, we have $A=X_j$, the operator of multiplication by x_j. If $a(x,\xi)=\xi_j$, we have $A=D_j=-i\partial/\partial x_j$.

We denote by $F[1,\infty]$ the set of all strictly increasing finite sequences of numbers in the interval $[1,\infty]$; $\{p_j\} \in F[1,\infty]$ means that there exist an integer $\ell \geqq 1$ and $\{p_j\}=\{p_1,\cdots,p_\ell\}$, $1 \leqq p_1 < \cdots < p_\ell \leqq \infty$. Given any $\{p_j\} \in F[1,\infty]$, we define the subspace $L^{\{p_j\}}(R^n\times R^n)$ of $\mathcal{S}'(R^n\times R^n)$ as $L^{\{p_j\}}(R^n\times R^n)=\sum_{p\in\{p_j\}} L^p(R^n\times R^n)$. We define also

$$(2.3)\qquad \|f\|_{L^{\{p_j\}}}=\inf_{\substack{f=\sum_{p\in\{p_j\}} f_p\\ f_p\in L^p(R^n\times R^n)}}\Big(\sum_{p\in\{p_j\}}(2\pi)^{n(1-1/p)}\|f_p\|_{L^p}\Big).$$

With the norm $\|\cdot\|_{L^{\{p_j\}}}$, the space $L^{\{p_j\}}(R^n\times R^n)$ forms a Banach space.

We denote by $B(L^2(R^n))$ the set of all bounded linear operators on $L^2(R^n)$ with the operator norm $\|\cdot\|$. By $C(L^2(R^n))$, we denote the set of all compact operators on $L^2(R^n)$. By $\|\cdot\|_p$ $(0<p\leqq\infty)$, we donote the various norms defined for compact operators T in terms of their characteristic numbers (i.e. the eigenvalues of $(T^*T)^{1/2}$, arranged in decreasing order and repeated according to multiplicity). By $C_p(L^2(R^n))$, we denote the set of all compact operators T such that $\|T\|_p$ is finite. In particular $C_1(L^2(R^n))$ is the trace class and $C_2(L^2(R^n))$ is the Hilbert-Schmidt class.

3. Results. 1) *Boundedness.* **Theorem 1** (cf. Cordes [3, Theorem D] and Kato [4, Theorem 5.2]). *Let $a\in\mathcal{S}'(R^n\times R^n)$ with $(1-\Delta_x)^s(1-\Delta_\xi)^t a\in L^{\{p_j\}}(R^n\times R^n)$ for some $s,t>n/4$ and $\{p_j\}\in F[1,\infty]$. Then $a(X,D)$ is L^2-bounded. Moreover*

$$(3.1)\qquad \|a(X,D)\|\leqq C_{n,s,t}\|(1-\Delta_x)^s(1-\Delta_\xi)^t a\|_{L^{\{p_j\}}},$$

where the constant $C_{n,s,t}$ is independent of a and $\{p_j\}$.

Theorem 2 (cf. Cordes [3, Theorem B_1'] and Kato [4, Theorem 5.1]). *Let $\{p_j\}\in F[1,\infty]$. If $D_x^\alpha D_\xi^\beta a\in L^{\{p_j\}}(R^n\times R^n)$ for $|\alpha|,|\beta|\leqq[n/2]+1$, then $a(X,D)$ is L^2-bounded. Moreover*

$$(3.2)\qquad \|a(X,D)\|\leqq C_n\sum_{|\alpha|,|\beta|\leqq[n/2]+1}\|D_x^\alpha D_\xi^\beta a\|_{L^{\{p_j\}}}.$$

Here and hereafter C_n denotes a positive constant independent of a and $\{p_j\}$.

Theorem 3 (cf. Kato [4, Theorem 5.3]). *Let $\{p_j\}\in F[1,\infty]$ and $0<\rho<1$. If $(1+|\xi|)^{(|\beta|-|\alpha|)\rho}D_x^\alpha D_\xi^\beta a\in L^{\{p_j\}}(R^n\times R^n)$ for $|\alpha|\leqq[n/2]+2$, $|\beta|\leqq[n/2]+1$, then $a(X,D)$ is L^2-bounded. Moreover*

$$(3.3)\qquad \|a(X,D)\|\leqq C_n\sum_{\substack{|\alpha|\leqq[n/2]+2\\|\beta|\leqq[n/2]+1}}\|(1+|\xi|)^{(|\beta|-|\alpha|)\rho}D_x^\alpha D_\xi^\beta a\|_{L^{\{p_j\}}}.$$

Theorem 3′. *Let $\{p_j\}\in F[1,\infty]$ and $0<\rho<1$. If*

$$(1+|x|)^{(|\alpha|-|\beta|)\rho}D_x^\alpha D_\xi^\beta a\in L^{\{p_j\}}(R^n\times R^n)$$
$$\text{for } |\alpha|\leqq[n/2]+1,\ \|\beta\|\leqq[n/2]+2,$$

then $a(X,D)$ is L^2-bounded. Moreover

$$(3.3')\qquad \|a(X,D)\|\leqq C_n\sum_{\substack{|\alpha|\leqq[n/2]+1\\|\beta|\leqq[n/2]+2}}\|(1+|x|)^{(|\alpha|-|\beta|)\rho}D_x^\alpha D_\xi^\beta a\|_{L^{\{p_j\}}}.$$

2) *Compactness.* Here we denote by $\chi_R(x,\xi)$ the characteristic function of the set $\{(x,\xi)\in R^n\times R^n;\ |x|^2+|\xi|^2>R^2\}$.

Theorem 4 (cf. Cordes [3, Theorem E]). *Let $\{p_j\}\in F[1,\infty]$. Assume $D_x^\alpha D_\xi^\beta a\in L^{\{p_j\}}(R^n\times R^n)$ and $\lim_{R\to\infty}\|\chi_R D_x^\alpha D_\xi^\beta a\|_{L^{\{p_j\}}}=0$ for $|\alpha|,|\beta|\leqq[n/2]+1$. Then $a(X,D)$ is a compact operator on $L^2(R^n)$.*

Theorem 5. *Let $\{p_j\}\in F[1,\infty]$ and $0<\rho<1$. Assume*

$$(1+|\xi|)^{(|\beta|-|\alpha|)\rho}D_x^\alpha D_\xi^\beta a\in L^{\{p_j\}}(R^n\times R^n)$$

and

$$\lim_{R\to\infty} \|\chi_R(1+|\xi|)^{(|\beta|-|\alpha|)\rho} D_x^\alpha D_\xi^\beta a\|_{L^{\{p_j\}}}=0 \quad \text{for } |\alpha|\leqq[n/2]+2,\ |\beta|\leqq[n/2]+1.$$

Then $a(X,D)$ is a compact operator on $L^2(R^n)$.

Theorem 5'. *Let $\{p_j\}\in F[1,\infty]$ and $0<\rho<1$. Assume*

$$(1+|x|)^{(|\alpha|-|\beta|)\rho} D_x^\alpha D_\xi^\beta a \in L^{\{p_j\}}(R^n\times R^n)$$

and

$$\lim_{R\to\infty} \|\chi_R(1+|x|)^{(|\alpha|-|\beta|)\rho} D_x^\alpha D_\xi^\beta a\|_{L^{\{p_j\}}}=0 \quad \text{for } |\alpha|\leqq[n/2]+1,\ |\beta|\leqq[n/2]+2.$$

Then $a(X,D)$ is a compact operator on $L^2(R^n)$.

Remark. We note that the following conditions are mutually equivalent for $f\in L^{\{p_j\}}(R^n\times R^n)$ with $\{p_j\}=\{p_1,\cdots,p_\ell\}\in F[1,\infty]$.

(i) $\lim_{R\to\infty}\|\chi_R f\|_{L^{\{p_j\}}}=0$.

(ii) There exist $f_j\in L^{p_j}(R^n\times R^n)$ $(j=1,\cdots,\ell)$ such that $f=\sum_{j=1}^{\ell} f_j$ and $\lim_{R\to\infty}\|\chi_R f_i\|_{L^{p_j}}=0$ $(j=1,\cdots,\ell)$.

(iii) If $\infty\in\{p_j\}$ (i.e. $p_\ell=\infty$), there exist $f_j\in L^{p_j}(R^n\times R^n)$ $(j=1,\cdots,\ell)$ such that $f=\sum_{j=1}^{\ell} f_j$ and $\lim_{R\to\infty}\|\chi_R f_\ell\|_{L^\infty}=0$.

4. Outlines of Proofs. 1) *Proof of Theorem* 1. A basic tool is the following identity formulated by Kato [4];

$$(4.1)\qquad (b*g)(X,D)=\iint_{R^n\times R^n} dxd\xi\, b(x,\xi)e^{i\xi X}e^{-ixD}g(X,D)e^{ixD}e^{-i\xi X},$$

where $*$ denotes the convolution on $R^n\times R^n$.

Lemma 1. *Let $1/p+1/q=1$, $1\leqq p\leqq\infty$. If $b\in L^p(R^n\times R^n)$ and $G\in C_q(L^2(R^n))$ then $b\{G\}\in B(L^2(R^n))$, where*

$$(4.2)\qquad b\{G\}=\iint_{R^n\times R^n} dxd\xi\, b(x,\xi)e^{i\xi X}e^{-ixD}Ge^{ixD}e^{-i\xi X}$$

as a strong (improper) integral. The mapping $b, G\mapsto b\{G\}$ has the following properties.

(i) $\|b\{G\}\|\leqq(2\pi)^{n/q}\|b\|_{L^p}\|G\|_q$.

(ii) $b\geqq 0$ *and* $G\geqq 0$ *imply* $b\{G\}\geqq 0$.

(iii) $|(b\{G\}u,v)_{L^2}|^2\leqq(|b|\{|G|\}u,u)_{L^2}(|b|\{|G^*|\}v,v)_{L^2}$ *for* $u,v\in L^2(R^n)$.

Here $G\geqq 0$ means that G is non-negative self-adjoint and $|G|$ means $(G^*G)^{1/2}$.

Let ψ_s be the unique solution within $\mathcal{S}'(R^n)$ for $(1-\Delta)^s\psi_s=\delta$, where s is a real number, Δ is the Laplacian, and δ is the delta function. It is well known that $\psi_s\in C^\infty(R^n\backslash\{0\})$,

$$(4.3)\qquad \begin{aligned} &D^\alpha\psi_s(x)=0(1+|x|^{2s-n-|\alpha|}) \text{ as } |x|\to 0 \text{ if } 2s-n-|\alpha|\neq 0,\\ &D^\alpha\psi_s(x) \text{ decays exponentially as } |x|\to\infty. \end{aligned}$$

Let $g(x,\xi)=\psi_s(x)\psi_t(\xi)$ with $s,t>n/4$ then $g(X,D)$ has an extension in $C_1(L^2(R^n))$ because of the following lemma.

Lemma 2. *Let $\phi,\varphi\in L^2(R^n)$ and $g(x,\xi)=\phi(x)\varphi(\xi)$. If ϕ and φ decay exponentially as $|x|\to\infty$, then $g(X,D)$ has an extension G in $C_1(L^2(R^n))$*

(i.e. $g(X,D)\subset G\in C_1(L^2(R^n))$.

Replacing b in (4.1) by $(1-\Delta_x)^s(1-\Delta_\xi)^t a$ and applying Lemma 1, we can prove Theorem 1. For Theorem 2, it suffices that the assumption implies that of Theorem 1. This assertion is obtained from the following as in Cordes [3].

Lemma 3. *For any $s>0$, we can write $(1-\Delta)^{1/2-s}=(1-\Delta)^{-(1/2+s)}-i\sum_{j=1}^n S_j^s D_j$, where $(1-\Delta)^{-(1/2+s)}$ and S_j^s have the L^1-convolution kernels $\psi_{1/2+s}$ and $\partial\psi_{1/2+s}/\partial x_j$ respectively.*

2) *Proof of Theorem* 3. In the same way as Kato [4], we use the partition of unity $\{\Phi_k(\xi)\}_{k=1}^\infty$ on R^n such that

$$|D_\xi^\beta \Phi_k(\xi)|\leqq C(1+|\xi|)^{-|\beta|\rho} \quad \text{for } |\beta|\leqq[n/2]+1, \tag{4.4}$$

$$||\xi|-k^{1/1-\rho}|\leqq Ck^{\rho/1-\rho} \quad \text{if } \xi\in \operatorname{supp}\Phi_k, \tag{4.5}$$

where C is a constant independent of k. Set

$$\begin{aligned} &a_k(x,\xi)=\Phi_k(\xi)a(x,\xi), \qquad k=1,2,3,\cdots, \text{ so that}\\ &a(x,\xi)=\sum_{k=1}^\infty a_k(x,\xi), \qquad a(X,D)=\sum_{k=1}^\infty a_k(X,D). \end{aligned} \tag{4.6}$$

In view of (4.4) and (4.5), there is a constant C independent of k such that

$$|(k^{-\gamma}D_x)^\alpha(k^\gamma D_\xi)^\beta a_k(x,\xi)|\leqq C\chi_k(\xi)\sum_{\beta'\leqq\beta} f_{\alpha,\beta'}(x,\xi) \tag{4.7}$$

for $|\alpha|\leqq[n/2]+2$, $|\beta|\leqq[n/2]+1$. Here $\gamma=\rho/1-\rho>0$, χ_k denotes the characteristic function of supp Φ_k,

$$f_{\alpha,\beta}(x,\xi)=|(1+|\xi|)^{(|\beta|-|\alpha|)\rho}D_x^\alpha D_\xi^\beta a(x,\xi)|. \tag{4.8}$$

From (4.7) and Theorem 2 and the assumption of Theorem 3, we obtain

$$a_k(X,D)\subset A_k\in C_1(L^2(R^n)). \tag{4.9}$$

The following lemma which follows from Lemma 3 is equivalent to Lemma 5.4 in Kato [4], but we need explicit formula for calculation.

Lemma 4. *There exist $\sigma>n/4+1/2$, $\tau>n/4$ and non-negative function $\mu\in L^1(R^n)$ such that $\mu(x)$ decays exponentially as $|x|\to\infty$ and*

$$|b_k(x,\xi)|\leqq\begin{cases}\displaystyle\int dy\, k^{n\gamma}\mu(k^\gamma(x-y))\chi_k(\xi)F(y,\xi) & \text{if } [n/2] \text{ is odd},\\ \displaystyle\int d\eta\, k^{-n\gamma}\mu(k^{-\gamma}(\xi-\eta))\chi_k(\eta)F(x,\eta) & \text{if } [n/2] \text{ is even},\end{cases} \tag{4.10}$$

where

$$b_k(x,\xi)=(1-k^{-2\gamma}\Delta_x)^\sigma(1-k^{2\gamma}\Delta_\xi)^\tau a(x,\xi), \tag{4.11}$$

$$F(x,\xi)=\sum_{\substack{|\alpha|\leqq[n/2]+2\\|\beta|\leqq[n/2]+1}}|f_{\alpha,\beta}(x,\xi)|, \tag{4.12}$$

and the constants σ,τ and the function μ do not depend on k but only on n.

With these numbers σ,τ, we define $g_k(x,\xi)=\psi_\sigma(k^\gamma x)\psi_\tau(k^{-\gamma}\xi)$. Then

$$g_k(X,D)\subset G_k\in C_1(L^2(R^n)) \quad \text{(from (4.3) and Lemma 2)}, \tag{4.13}$$

$$A_k=b_k\{G_k\} \quad \text{(from (4.1) and (4.2))}, \tag{4.14}$$

$$|(A_k u,v)_{L^2}|^2\leqq(|b_k|\{|G_k|\}u,u)_{L^2}(|b_k|\{|G_k{}^*|\}v,v)_{L^2} \tag{4.15}$$

for $u, v \in L^2(R^n)$ (from Lemma 1 (iii)).

Taking (4.6) into account, we must prove the boundedness of $\sum_k A_k$. To do this, it is enough to show that $\sum_k |b_k| \{|G_k|\}$ and $\sum_k |b_k| \{|G_k^*|\}$ converge with respect to the operator norm.

Let $\{p_j\}=\{p_1, \cdots, p_\ell\} \in F[1, \infty]$. From the assumption of Theorem 3, each $f_{\alpha,\beta}(x, \xi)$ of (4.8) can be written as

$$f_{\alpha,\beta}(x, \xi)=\sum_{j=1}^{\ell} f_{\alpha,\beta,j}(x, \xi), \qquad f_{\alpha,\beta,j} \in L^{p_j}(R^n \times R^n). \tag{4.16}$$

We put

$$f_j(x, \xi)=\sum_{\substack{|\alpha|\leqq[n/2]+2 \\ |\beta|\leqq[n/2]+1}} |f_{\alpha,\beta,j}(x, \xi)| \in L^{p_j}(R^n \times R^n). \tag{4.17}$$

From (4.10) and (4.12), we obtain that

$$\sum_{k=1}^{\infty} (|b_k| \{|G_k|\}u, u) \leqq \begin{cases} \sum_{j=1}^{\ell} P_j & \text{if } [n/2] \text{ is odd,} \\ \sum_{j=1}^{\ell} Q_j & \text{if } [n/2] \text{ is even,} \end{cases} \tag{4.18}$$

where

$$P_j=\sum_{k=1}^{\infty} \iint dx d\xi \int dy \mu(k^r(x-y)) k^{nr} \chi_k(\xi) f_j(y, \xi) \times (e^{i\xi X} e^{-ixD} |G_k| e^{ixD} e^{-i\xi X} u, u), \tag{4.19}$$

$$Q_j=\sum_{k=1}^{\infty} \iint dx d\xi \int d\eta \mu(k^{-r}(\xi-\eta)) k^{-nr} \chi_k(\eta) f_j(x, \eta) \times (e^{i\xi X} e^{-ixD} |G_k| e^{ixD} e^{-i\xi X} u, u). \tag{4.20}$$

In the same way we obtain that

$$\sum_{k=1}^{\infty} (|b_k| \{|G_k^*|\}u, u) \leqq \begin{cases} \sum_{j=1}^{\ell} P_j^* & \text{if } [n/2] \text{ is odd,} \\ \sum_{j=1}^{\ell} Q_j^* & \text{if } [n/2] \text{ is even,} \end{cases} \tag{4.18'}$$

where P_j^* and Q_j^* are defined by replacing $|G_k|$ by $|G_k^*|$ in (4.19) and (4.20).

In the case $p_j=1$, we can easily obtain that

$$P_j, P_j^*, Q_j, Q_j^* \leqq C_n \|G\| \cdot \|f_j\|_{L^1} \|u\|^2. \tag{4.21}$$

In the case $p_j=\infty$, we notice that G_1 and G_k $(k \geqq 2)$ are unitary equivalent and that $\sum_{k=1}^{\infty} \chi_k(\xi-k^r\eta) \leqq 2(C+|\eta|)$ for $\xi, \eta \in R^n$ (equivalent to Lemma 5.5 in Kato [4]).

Lemma 5 (cf. Kato [4, Lemma 4.2]). *Let $g(x, \xi)=\psi_s(x)\psi_t(\xi)$. Then the followings are valid.*

i) *$D_j g(X, D)$, $g(X, D)D_j$, $|D| g(X, D)$ and $g(X, D)|D|$ have extension in $C_1(L^2(R^n))$ if $s>n/4+1/2$, $t>n/4$.*

ii) *$g(X, D)X_j$, $X_j g(X, D)$, $g(X, D)|X|$ and $|X| g(X, D)$ have extension in $C_1(L^2(R^n))$ if $s>n/4$, $t>n/4+1/2$.*

Using above Lemma 5, we obtain as in Kato [4] that

$$P_j, P_j^*, Q_j, Q_j^* \leqq C_n(\|G\|_1+\|G|D|\|_1+\||D|G\|_1) \|f_j\|_{L^\infty} \|u\|^2. \tag{4.22}$$

In the case $1<p_j<\infty$, we use the Hölder inequality and obtain that for any constant $C>0$

$$\begin{aligned}(4.23)\qquad &\int dy\mu(k^r(x-y))k^{nr}\chi_k(\xi)f_j(y,\xi)\\ &\quad\leqq (C^{p_j}/p_j)\int dy\mu(k^r(x-y))k^{nr}\chi_k(\xi)f_j^{p_j}(y,\xi)\\ &\qquad+(C^{-q_j}/q_j)\int dy\mu(k^r(x-y))k^{nr}\chi_k(\xi),\end{aligned}$$

$$\begin{aligned}(4.23')\qquad &\int d\eta\mu(k^{-r}(\xi-\eta))k^{-nr}\chi_k(\eta)f_j(x,\eta)\\ &\quad\leqq (C^{p_j}/p_j)\int d\eta\mu(k^{-r}(\xi-\eta))k^{-nr}\chi_k(\eta)f_j^{p_j}(x,\eta)\\ &\qquad+(C^{-q_j}/q_j)\int d\eta\mu(k^{-r}(\xi-\eta))k^{-nr}\chi_k(\eta).\end{aligned}$$

Using (4.23) and (4.23′), we can reduce the estimates of P_j, P_j^*, Q_j, and Q_j^* in this case to those in the case $p_j=1$ and in the case $p_j=\infty$. Choosing a suitable constant C, we obtain that

$$(4.24)\qquad P_j, P_j^*, Q_j, Q_j^* \leqq C_n(\|G\|_1+\|G|D|\|_1+\||D|G\|_1)\|f_j\|_{L^{p_j}}\|u\|^2.$$

Noting the symmetry of roles of (x, X) and (ξ, D), we can similarly prove Theorem 3′. We can prove as in Cordes [3] that Theorems 4, 5, and 5′ follow from Theorems 2, 3, and 3′ respectively.

References

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