34. On Kodaira Dimension of Graphs

By Shigeru IITAKA
Department of Mathematics, University of Tokyo, Hongo, Tokyo

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1. We shall study curves on a complete non-singular rational surface $\bar{S}$ defined over the field of complex numbers.

Let D be a reduced divisor consisting of rational curves $C_1, \cdots, C_s$. We let C_i have at most normal crossings and suppose that the singularity of D is ordinary, i.e., for any $p \in D$, if we take all components $C_1, \cdots, C_r$ passing through p, then all tangents to $C_1, \cdots, C_r$ at p are mutually distinct.

With each such D, we associate the graph $\Gamma(D)$. Now, define the following numerical invariants of a graph Γ:

$P_m(\Gamma) = \mathrm{Min}\,\{\bar{P}_m(\bar{S}-D)\,;\, \bar{S}$ is a complete rational surface and $\Gamma = \Gamma(D)\}$,

$\kappa(\Gamma) = \mathrm{Inf}\,\{\kappa(\bar{S}-D)\,;$ the same as above$\}$.

Here, $\bar{P}_m(S)$ denotes the logarithmic m-genus of S and $\bar{\kappa}(S)$ the logarithmic Kodaira dimension of S (see [1] and [2]).

Let C_1 be an edge-component (i.e., $(C_1, D^0)=1$ when $D=C_1+D^0$) of a divisor D corresponding to a graph Γ. Removing C_1 we have a new graph Γ_1. Now, choose a surface $\bar{S}_1$ and boundary D_1 such that $\Gamma_1 = \Gamma(D_1)$ and $P_m(\Gamma_1) = \bar{P}_m(\bar{S}-D_1)$ for any fixed m. Blow up at p_1, i.e., $\mu: \bar{S} = Q_{p_1}(\bar{S}_1) \to \bar{S}_1$ and put $D' + \mu^{-1}(p) = \mu^{-1}(D_1)$. Then

$$\bar{P}_m(\bar{S}-\mu^{-1}(D_1)) = \bar{P}_m(\bar{S}_1 - D_1),$$
$$\bar{P}_m(\bar{S}-\mu^{-1}(D_1)) \geqq \bar{P}_m(\bar{S}-D').$$

Since $\Gamma = \Gamma(\mu^{-1}(D_1))$ and $\Gamma_1 = \Gamma(D')$, we get

$$P_m(\Gamma_1) = \bar{P}_m(\bar{S}_1 - D_1) \geqq \bar{P}_m(\bar{S}-D').$$

By definition, $\bar{P}_m(\bar{S}-D') \geqq P_m(\Gamma_1)$. Thus

Proposition 1. $P_m(\Gamma) = P_m(\Gamma_1)$.

Similarly, one obtains

$$\kappa(\Gamma) = \kappa(\Gamma_1).$$

Hence, Γ may be assumed to have no edge-components and no isolated edges.

In view of $\bar{p}_g$-formula [2], we obtain

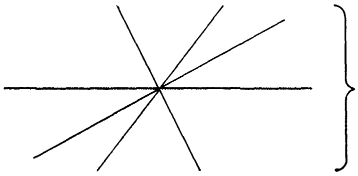
Propositoin 2. $P_1(\Gamma) = \bar{p}_g(\bar{S}-D) = h(\Gamma(D))$.

Here, $\bar{S}$ is a complete rational surface and D is a reduced divisor such that $\Gamma = \Gamma(D)$. Moreover, $h(\Gamma)$ denotes the cyclotomic number of Γ.

2. In this section, we restrict ourselves to the graphs Γ with $P_1(\Gamma) = h(\Gamma) = 0$.

Theorem 1. $\kappa(\Gamma) = -\infty$ *if and only if* Γ *is a graph of type* A_m.

A_m :

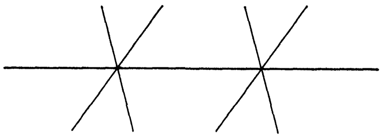


m lines

Moreover, $P_2(\Gamma)=0$ *implies* $\kappa(\Gamma)=-\infty$.

Theorem 2. $\kappa(\Gamma)=0$ *if and only if* Γ *is of the following type*:

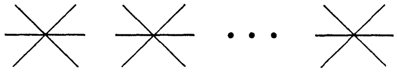
G_2 :



Moreover, $P_2(\Gamma)=P_4(\Gamma)=1$ *implies* $\kappa(\Gamma)=0$.

Theorem 3. *If* $\kappa(\Gamma)=1$, *then* Γ *is of type* G'_n $(6\geqq n\geqq 2)$.

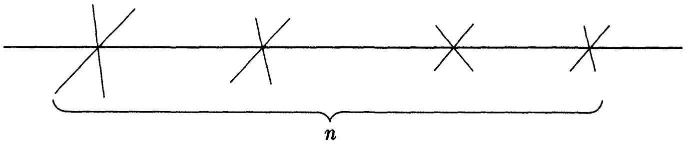
G'_n:



These are derived from the following lemmas.

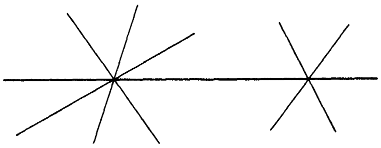
Lemma 1. *Let* Γ *be a graph of type* G_n $(n\geqq 2)$. *Then* $P_2(\Gamma)\geqq n-1$ *and hence* $\kappa(\Gamma)\geqq 0$. *Moreover, if* $n\geqq 3$, *then* $\kappa(\Gamma)=2$.

Here, by G_n we denote the following graph.



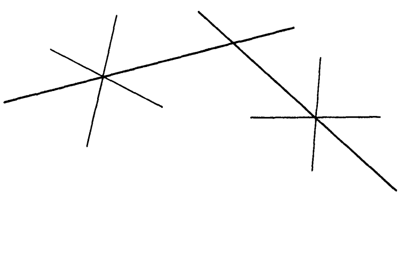
Lemma 2. *If* Γ *is of type* G'_n, *then* $P_2(\Gamma)=n-1$, $P_3(\Gamma)\geqq 2$. *Moreover,* $\kappa(\Gamma)=1$ *if and only if* $2\leqq n\leqq 6$.

Lemma 3. *If* Γ *is of the following type*:



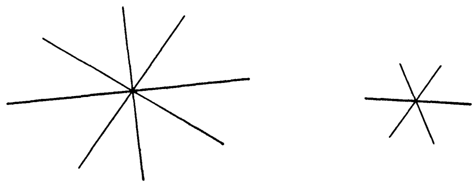
then $P_2(\Gamma)\geqq 2$ *and* $\kappa(\Gamma)=2$.

Lemma 4. *If* Γ *is of the following type*:



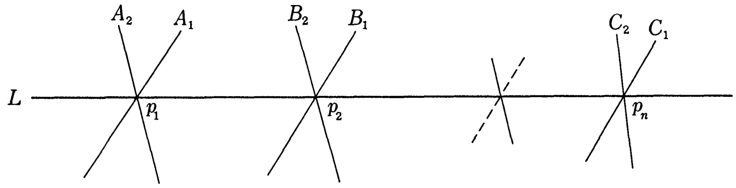
then $P_3(\Gamma)\geqq 2$ *and* $\kappa(\Gamma)=2$.

Lemma 5. *If* Γ *is of the following type:*



then $P_2(\Gamma)\geqq 2$ *and* $\kappa(\Gamma)=2$.

Proofs of Lemmas 1 and 4. First we take a reduced divisor as follows:



Consider a composition $\mu: \bar{S}^{\#}\to\bar{S}$ of blowing ups at $p_1, \cdots, p_n$. Then, letting $E_j=\mu^{-1}(p_j)$ we have

$$K(\bar{S}^{\#})+\mu^{-1}(D)=\mu^*(K(\bar{S})+D)-\sum E_j.$$

By Riemann Roch theorem,

$$\dim|K(\bar{S})+A_1+A_2+L|=\dim|K(\bar{S})+B_1+B_2+L|=0.$$

Hence, let $X\in|K(\bar{S})+A_1+A_2+L|$ and $Y\in|K(\bar{S})+B_1+B_2+L|$. Then

$$2(K(\bar{S}^{\#})+\mu^{-1}(D))\sim X+Y+A_1'+A_2'+B_1'+B_2'+\cdots+C_1'+C_2'+C_1+C_2$$
$$\geqq Y+\cdots+C_1+C_2\sim K(\bar{S})+B_1+B_2+\cdots+C_1+C_2+L.$$

(A' denotes the proper transform of A.)

Applying Riemann Roch theorem, we have

$$\dim|K(\bar{S})+B_1+B_2+\cdots+C_1+C_2+L|+1$$
$$=\pi(B_1+B_2+\cdots+C_1+C_2+L)=n-1.$$

Here, $\pi(D)$ denotes the virtual genus of D.

Assuming $\bar{\kappa}(\bar{S}-D)=1$, we consider a logarithmic canonical fibered surface $\varphi: \bar{S}\to J\rightleftarrows \boldsymbol{P}^1$ of S. Take a general fiber $\Gamma_u=\varphi^{-1}(u)$. Then $(K(\bar{S}^{\#})+\mu^{-1}(D), \Gamma_u)=\Gamma_u^2=\pi(\Gamma_u)=0$. Hence, using the explicit formula for $2(K(\bar{S}^{\#})+\mu^{-1}(D))$, we derive $(C_1+C_2, \Gamma_u)=0$ and $(L^*, \Gamma_u)=2$, L^* being the proper transform of Γ. Hence $(C_1', \Gamma_u)=(C_2', \Gamma_u)=(G, \Gamma_u)=0$. Furthermore, $(A_1', \Gamma_u)=(A_2', \Gamma_u)=(E, \Gamma_u)=(B_1', \Gamma_u)=(B_2', \Gamma_u)=0$. Hence, $A_1'+A_2'+E$ is a part of a fiber $\varphi^{-1}(a)$. Similarly, $B_1'+B_2'+E\subseteq\varphi^{-1}(b)$, $C_1'+C_2'+G\subseteq\varphi^{-1}(c)$. Let $\psi=\varphi|\Gamma^*: \Gamma^*\to J$, which is 2-sheeted, and which ramifies at $E\cap\Gamma^*$, $F\cap\Gamma^*$, $G\cap\Gamma^*$. This contradicts the Hurwitz formula for ψ.

Example 1. Let $H_1, \cdots, H_5$ be 5 lines in $\boldsymbol{P}^2$ as in Fig. 1. Blow-

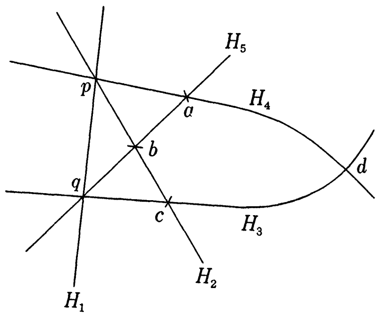
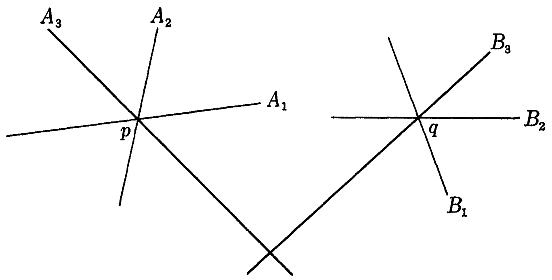


Fig. 1

ing up at a, b, c and d, we have a birational morphism $\rho: \bar{S} \to \boldsymbol{P}^2$ and put $D = \rho^{-1}(H_1 + \cdots + H_5) - \rho^{-1}(a) - \rho^{-1}(b) - \rho^{-1}(c)$ (as a divisor). Then $\Gamma(D)$ is of type G_2 and $\bar{P}_{2m}(\bar{S} - D) = 1$ for any $m \geqq 1$. Hence $\bar{\kappa}(\bar{S} - D) = 0$.

Now, we come back to the proof of Lemma 4 and take a reduced divisor D as follows:



Blowing up at p and q, we have a proper birational morphism $\mu: \bar{S}^\# \to \bar{S}$. Defining $E = \mu^{-1}(p)$ and $F = \mu^{-1}(q)$, we have

$$K(\bar{S}^\#) + \mu^{-1}(D) = \mu^*(K(\bar{S}) + A_1 + A_2 + A_3 + B_1 + B_2 + B_3) - E - F.$$

Take $X \in |K(\bar{S}) + A_1 + A_2 + A_3|$ and $Y \in |K(\bar{S}) + B_1 + B_2 + B_3|$. Then

$$\begin{aligned} 3(K(\bar{S}^\#) + \mu^{-1}(D)) &\sim X + Y + A_1' + A_2' + A_3' + B_1' + B_2' + B_3' \\ &\quad + K(\bar{S}) + A_1 + A_2 + A_3 + B_1 + B_2 + B_3 \\ &\geqq K(\bar{S}) + A_1 + A_2 + A_3 + B_1 + B_2 + B_3. \end{aligned}$$

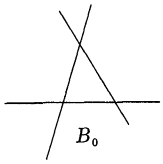
Hence $\bar{P}_3(\bar{S} - D) \geqq \pi(A_1 + A_2 + A_3 + B_1 + B_2 + B_3) = 2$. Furthermore, we find an effective divisor Z such that

$$6(K(\bar{S}^\#) + \mu^{-1}(D)) \sim Z \geqq A_1 + A_2 + A_3 + B_1 + B_2 + B_3 = D.$$

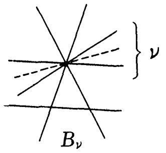
From this it follows that $\bar{\kappa}(\bar{S} - D) = 2$. We omit the detail.

3. The case in which $h(\Gamma) > 0$ is more complicated.

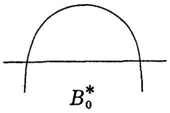
Theorem 4. *If $\bar{P}_1(\Gamma) = 1$ and $\kappa(\Gamma) = 0$, then Γ is one of the following types:*



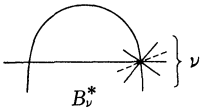
B_0



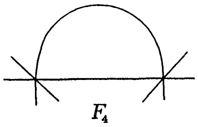
B_ν



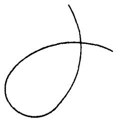
B_0^*



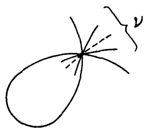
B_ν^*



F_4



B_0^{**}



B_ν^{**}

Moreover, $\bar{P}_1(\Gamma)=P_4(\Gamma)=1$ *yields* $\bar{\kappa}(\Gamma)=0$.

Theorem 5. *If* $\kappa(\Gamma)=1$, Γ *is classified into the following types* $C_n, C'_n, C''_n, D^I_n, D^{I*}_n, \cdots, D^{II**}_n, X_{l,m,n,k}, Y_{l,m,n}$.

Details will appear elsewhere.

References

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[4] I. Wakabayashi: On Kodaira dimension of complements of plane curves (to appear).