33. G-Manifolds and G-Vector Fields with Isolated Zeros

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Let G be a finite group. A G-manifold is a smooth manifold M together with a smooth G-action on M, and a (continuous) G-vector field on a G-manifold M is a continuous G-equivariant cross section of the tangent bundle $\tau(M)$ of M. The object of this paper is to apply the equivariant homotopy theory of representation spheres [4] to remove isolated zeros of G-vector fields.

1. Preliminaries. Let M be a G-manifold. For any $x\in M$, G_x denotes the isotropy subgroup at x. For any subgroup H of G, define $M_H=\{x\in M\mid G_x=H\}$ and $M^H=\{x\in M\mid H\subset G_x\}$. Then M_H and M^H are submanifolds of M. Let $s: M\to\tau(M)$ be a G-vector field on M. s induces a vector field $s^H: M^H\to\tau(M^H)$ on M^H by restricting s on M^H.

Recall the index of a vector field s on M at an isolated zero $z\in M-\partial M$. The index is denoted by $\operatorname{ind}(z; s)$, and defined to be the degree of the map

$$f=\frac{d\varphi\circ s\circ\varphi^{-1}}{\|d\varphi\circ s\circ\varphi^{-1}\|}: S^{n-1}\to S^{n-1},$$

where φ is a chart from a small neighborhood of z into $\boldsymbol{R}^n$ taking z to 0, and $n=\dim M$. The map f describes the behavior of s near z. When M is a G-manifold and s is a G-vector field, we may take φ so as to be a G_z-equivariant chart from a G_z-invariant neighborhood of z into an orthogonal representation V of G_z taking z to 0. Moreover, the map f is a G_z-equivariant map from $S(V)$ to itself, where $S(V)$ is the unit sphere in V. For any subgroup H of G_z, z is also an isolated zero of s^H, and we see $\operatorname{ind}(z; s^H)=\deg f^H$, where $f^H: S(V)^H\to S(V)^H$ is the restriction of f on $S(V)^H$.

Convention. For the only map $f:\phi\to\phi$ of an empty set, define $\deg f=1$. So the index of a vector field on a 0-dim manifold at each point is 1. For a map $f: S^0\to S^0$, define $\deg f=1$ if f is the identity, $\deg f=0$ if f maps S^0 to one point, and $\deg f=-1$ if f interchanges the two points of S^0.

2. Removing zeros. Theorem 1. *Let G be a finite abelian group, and K a subgroup of G. Let s be a G-vector field on a G-manifold M. Let A be a connected component of M_K, and $\{z_1, z_2, \cdots, z_p\}$ the zeros of s on A. Assume that all z_i's are isolated zeros of s and are off ∂M, and assume that for any subgroup H of K,*

$$\sum_{i=1}^{p} \operatorname{ind}(z_i\,;\, s^H)=0.$$

Then for any G-invariant neighborhood U of G(A) in M, we obtain a G-vector field t on M which has no more zero on G(A) and agrees with s on M−U and on ∂M.

To prove the theorem we need the following two lemmas.

Lemma 1. *Let M be a G-manifold. (In this lemma G may be any compact Lie group.) Let* $x, y \in \operatorname{Int} M$ *be points in a connected component A of* M_H *for a subgroup H of G. Then there exists a G-equivariant isotopy* $F: M\times I \to M$ *with* F_0 *the identity and with* $F_1(G(x)) = G(y)$. *Moreover, F can be taken to be constant in* $t \in I$ *outside a given G-invariant neighborhood in M of some compact subset in G(A).*

Proof. Construct a G-equivariant isotopy $f: G(x)\times I\to M$ with f_0 the inclusion and with $f_1(G(x))=G(y)$. By means of an equivariant analogy of the isotopy extension theorem, extend f to the desired G-equivariant isotopy F of M.

Lemma 2. *Let G be a finite abelian group, and V an orthogonal representation of G containing trivial action. Let S(V) and D(V) be the unit sphere and the unit disc in V, respectively. Then a G-equivariant map* $f: S(V)\to S(V)$ *can be G-equivariantly extended over D(V) if and only if* $\deg f^H=0$ *for any subgroup H of G.*

This lemma follows from the classification theorem of equivariant homotopy classes of equivariant maps of representation spheres [4].

Proof of Theorem 1. For some orthogonal representation V of K with $\dim V=\dim M$, let $D(V)$ be a small disc which is K-equivariantly embedded in M and which is centered at a point in A. Assume that $D(V)$ is so small that

(i) $g(D(V))\cap D(V)=\phi$ for any $g\in G-K$,

(ii) $D(V)$ is contained in U, and

(iii) s has no zero on $D(V)-D(V)\cap A$.

We may use isotopies in Lemma 1 to push all zeros on $G(A)$ into $G(\operatorname{Int} D(V))$. Precisely, there exists a G-equivariant diffeomorphism α of M such that $\alpha(G(\{z_1,\cdots,z_p\}))\subset G(\operatorname{Int} D(V))$ and $\alpha=$identity on $M-U$ and on ∂M. Consider a G-vector field $s_1=d\alpha\circ s\circ\alpha^{-1}$. The zeros of s_1 on $G(A)$ are $G(\{\alpha(z_1),\cdots,\alpha(z_p)\})$ which are contained in $G(\operatorname{Int} D(V))$, and s_1 agrees with s on $M-U$ and on ∂M. Let $\{x_1,\cdots,x_q\}=\{z_i \mid \alpha(z_i)\in D(V)\}$. Then $p=aq$ for some integer $a>0$, and for any subgroup H of K

$$\begin{aligned}\sum_{i=1}^{q} \operatorname{ind}(\alpha(x_i)\,;\, s_1^H) &= \sum_{i=1}^{q} \operatorname{ind}(x_i\,;\, s^H)\\ &=1/a \sum_{i=1}^{p} \operatorname{ind}(z_i\,;\, s^H)\\ &=0.\end{aligned}$$

Since s_1 has no zero on $S(V)=\partial D(V)$, s_1 induces a K-equivariant map $f: S(V)\to S(V)$ which describes the behavior of s_1 on $S(V)$. We see that for any subgroup H of K

$$\deg f^H = \sum_{i=1}^{q} \operatorname{ind}(\alpha(x_i); s_1^H) = 0.$$

Then Lemma 2 implies that f extends to a K-equivariant map $f_1: D(V) \to S(V)$. (We note that the assumption on the indices of the zeros of s on A implies $\dim A > 0$, and that V contains trivial action.) f_1 induces a G-vector field on $G(D(V))$ which has no zero and agrees with s_1 on $G(S(V))$. So we obtain a G-vector field t on M which has no zero on $G(D(V))$ and agrees with s_1 outside $G(\operatorname{Int} D(V))$. t is a required G-vector field on M.

3. Existence of G-vector fields with finite zeros.

Theorem 2. *Let G be a finite group. Then any compact G-manifold M has a G-vector field s such that*

(i) *s has only finite zeros,*

(ii) *at all boundary points s is not zero and points inward, and*

(iii) *if z is a zero of s and if $K = G_z$, then* $\operatorname{ind}(z; s^H) = \operatorname{ind}(z; s^K)$ *for any subgroup H of K.*

We may construct such a G-vector field by the same method developed in [1] and [2]. So we omit the proof.

4. Application. As an application of our result we obtain

Theorem 3. *Let G be a finite abelian group of odd order. Let W be an n-dim compact G-manifold with $\partial W = M_0 \cup M_1$, where M_0 and M_1 are disjoint and are G-invariant $(n-1)$-dim closed submanifolds of ∂W. Then there exists a non-singular G-vector field on W which points inward on M_0 and outward on M_1 if and only if, for any subgroup H of G and for any connected component B of W^H,*

$$\chi(B) = \chi(B \cap M_0) = \chi(B \cap M_1),$$

where $\chi(\text{-})$ denotes Euler characteristic.

Note. Theorem 3 supplies a necessary and sufficient condition for M_0 and M_1 to be G-equivariantly Reinhart cobordant. See [3] for (non-equivariant) Reinhart cobordism. Also see [5] for $\boldsymbol{Z}_2$-equivariant Reinhart cobordism.

Proof of Theorem 3. To prove the necessity of the condition, let s be a non-singular G-vector field on W, and assume s points inward on M_0 and outward on M_1. For any H and B, $s^H | B$ is a non-singular vector field on B and points inward on $\partial B \cap M_0$ and outward on $\partial B \cap M_1$. Then $\chi(B) = \chi(B \cap M_0) = \chi(B \cap M_1)$ follows from [3].

Next to prove the sufficiency, let $P = M_0 \times [0,1]$ be a G-equivariant collar of M_0 in W, and let $Q = W - M_0 \times [0,1)$. By Theorem 2, there exist G-vector fields s_1 on P and s_2 on Q such that

(i) s_i $(i=1.2)$ has finite zeros,

(ii) s_1 points inward on ∂P and s_2 points outward on ∂Q, and

(iii) if z is a zero of s_i and if $K = G_z$,

then $\operatorname{ind}(z; s_i^H) = \operatorname{ind}(z; s_i^K)$ for any subgroup H of K. (Note: Theorem 2 implies at once $\operatorname{ind}(z; (-s_2)^H) = \operatorname{ind}(z; (-s_2)^K)$. However, in our situation where G is a finite abelian group of odd order, $\mathrm{ind}(z; (-s_2)^H) = \mathrm{ind}(z; (-s_2)^K)$ implies $\mathrm{ind}(z; s_2^H) = \mathrm{ind}(z; s_2^K)$.) s_1 and s_2 induces a G-vector field s on W which points inward on M_0 and outward on M_1. Moreover, for any subgroup K of G and for any connected component A of W_K, we may show that if $\chi(B) = \chi(B \cap M_0) = \chi(B \cap M_1)$ for any $B \subset W^H$ then the zeros of s on A satisfy the assumption in Theorem 1. Then we obtain a non-singular G-vector field on W which points inward on M_0 and outward on M_1.

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