32. *Cosine Families and Weak Solutions of Second Order Differential Equations*

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(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1978)

Let A be a densely defined closed linear operator on a real or complex Banach space X, let $T>0$ and let $g \in L^1(0, T; X)$. Recently, Ball [1] proved that there exists for each $x \in X$ a unique weak solution, suitable defined, of the equation $u'(t)=Au(t)+g(t)$, $t \in [0, T]$, $u(0)=x$ if and only if A is the infinitesimal generator of a (C_0)-semigroup $\{T(t); t \geq 0\}$ on X, and in this case the solution $u(t)$ is given by

$$u(t)=T(t)x+\int_0^t T(t-s)g(s)ds, \qquad t \in [0, T].$$

The purpose of this note is to establish the parallel relationship between cosine families and second order differential equations

$$(\mathrm{IV}; x, y) \qquad \begin{cases} w''(t)=Aw(t)+g(t), & t \in [0, T], \\ w(0)=x \in X, & w'(0)=y \in X. \end{cases}$$

Let A^* denote the adjoint of A and $\langle \ , \ \rangle$ the pairing between X and its dual space X^*.

Definition. A function $w \in C([0, T]; X)$ is a weak solution of (IV; x, y) if and only if for every $v \in D(A^*)$ the function $\langle w(t), v\rangle$ is differentiable on $[0, T]$, $(d/dt)\langle w(t), v\rangle$ is absolutely continuous on $[0, T]$ and

$$(1) \qquad \begin{cases} (d^2/dt^2)\langle w(t), v\rangle=\langle w(t), A^*v\rangle+\langle g(t), v\rangle & \text{a.e.} \ t \in [0, T], \\ w(0)=x \quad \text{and} \quad (d/dt)\langle w(t), v\rangle|_{t=0}=\langle y, v\rangle. \end{cases}$$

Our theorem is now stated as follows:

Theorem. *There exists for each pair* $[x, y] \in X \times X$ *a unique weak solution* $w(t)$ *of* (IV; x, y) *if and only if* A *is the infinitesimal generator of a cosine family* $\{C(t); t \in R=(-\infty, \infty)\}$ *on* X, *and in this case* $w(t)$ *is given by*

$$(2) \qquad w(t)=C(t)x+S(t)y+\int_0^t S(t-s)g(s)ds, \qquad t \in [0, T],$$

where $\{S(t); t \in R\}$ *is the sine family associated with* $\{C(t); t \in R\}$.

Remark. Let $B(X)$ denote the set of all bounded linear operators from X into itself. A one-parameter family $\{C(t); t \in R\}$ in $B(X)$ is called a *cosine family* if it satisfies the following conditions:

(i) $C(s+t)+C(s-t)=2C(s)C(t)$ for all $s, t \in R$;

(ii) $C(0)=I$ (the identity operator);

(iii) $C(t)x: R \to X$ is continuous for every $x \in X$.

The associated *sine family* $\{S(t); t \in R\}$ is the one-parameter family in $B(X)$ defined by

$$S(t)x = \int_0^t C(s)x\, ds \qquad \text{for } x \in X \text{ and } t \in R.$$

The *infinitesimal generator* A' of a cosine family $\{C(t); t \in R\}$ on X is defined by $A'x = \lim_{t\to 0} 2t^{-2}(C(t)x - x)$ whenever the limit exists. Let $\{C(t); t \in R\}$ be a cosine family on X, with the infinitesimal generator A' and the associated sine family $\{S(t); t \in R\}$; the following properties are well known (see [2], [4] and [5]):

(iv) there exist constants $K \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq Ke^{\omega|t|}$ for all $t \in R$,

(v) A' is a densely defined closed linear operator,

(vi) $\int_s^t S(\sigma)x\, d\sigma \in D(A')$ and $C(t)x - C(s)x = A' \int_s^t S(\sigma)x\, d\sigma$ for $x \in X$ and $s, t \in R$,

(vii) if $z(t): [0, T] \to X$ is twice (strongly) continuously differentiable, $z(t) \in D(A')$ for $t \in [0, T]$, and $(d^2/dt^2)z(t) = A'z(t)$ for $t \in [0, T]$ and $z(0) = z'(0) = 0$, then $z(t) = 0$ for all $t \in [0, T]$.

To prove the theorem we use the following lemmas (see [1, Lemma] and [3, (1.3.3.) Lemma]):

Lemma 1 ([1]). *Let $x, z \in X$ satisfy $\langle z, v\rangle = \langle x, A^*v\rangle$ for all $v \in D(A^*)$. Then $x \in D(A)$ and $z = Ax$.*

Lemma 2 ([3]). *Let $\{C(t); t \in R\}$ be a one-parameter family in $B(X)$ such that $C(t)x: R \to X$ is continuous for every $x \in X$. If*

(a) *$C(t)D(A) \subset D(A)$ for all $t \in R$,*

(b) *for each $x \in D(A)$, the function $C(t)x: R \to X$ is twice (strongly) continuously differentiable, and $C''(t)x = AC(t)x = C(t)Ax$ for $t \in R$, $C(0)x = x$ and $C'(0)x = 0$, then $\{C(t); t \in R\}$ is a cosine family on X and A is its infinitesimal generator.*

Proof of Theorem. Assume that A is the infinitesimal generator of a cosine family $\{C(t); t \in R\}$ on X. Let $x, y \in X$ and let w be given by (2). It is easily shown that $w \in C([0, T]; X)$. We want to show that w is a weak solution of (IV; x, y). Let $v \in D(A^*)$. By (vi), for every $t \in [0, T]$

$$\begin{aligned}\langle h^{-1}(C(t+h)x - C(t)x), v\rangle &= \left\langle A\left(h^{-1}\int_t^{t+h} S(s)x\, ds\right), v\right\rangle \\ &= \left\langle h^{-1}\int_t^{t+h} S(s)x\, ds, A^*v\right\rangle \to \langle S(t)x, A^*v\rangle \\ &\qquad\qquad \text{as } h \to 0,\end{aligned}$$

i.e., $(d/dt)\langle C(t)x, v\rangle = \langle S(t)x, A^*v\rangle$. Noting $(d/dt)\int_0^t S(t-s)g(s)ds = \int_0^t C(t-s)g(s)ds$, we have

$$(d/dt)\langle w(t), v\rangle = \langle S(t)x, A^*v\rangle + \langle C(t)y, v\rangle + \int_0^t \langle C(t-s)g(s), v\rangle ds$$

for $t \in [0, T]$. Since $C(t-s)g(s) = g(s) + A\int_0^{t-s} S(\sigma)g(s)d\sigma$ by (vi),

$$(d/dt)\langle w(t), v\rangle = \langle S(t)x, A^*v\rangle + \langle C(t)y, v\rangle + \int_0^t \langle g(s), v\rangle ds + \int_0^t \left[\int_s^t \langle S(\sigma - s)g(s), A^*v\rangle d\sigma\right] ds$$
$$= \langle S(t)x, A^*v\rangle + \langle C(t)y, v\rangle + \int_0^t \left[\int_0^\sigma \langle S(\sigma - s)g(s), A^*v\rangle ds\right] d\sigma + \int_s^t \langle g(s), v\rangle ds$$

for $t \in [0, T]$. This implies that $(d/dt)\langle w(t), v\rangle$ is absolutely continuous and

$$(d^2/dt^2)\langle w(t), v\rangle = \langle C(t)x, A^*v\rangle + \langle S(t)y, A^*v\rangle + \int_0^t \langle S(t-s)g(s), A^*v\rangle ds + \langle g(t), v\rangle$$
$$= \langle w(t), A^*v\rangle + \langle g(t), v\rangle \qquad \text{for a.e.} t \in [0, T].$$

Moreover $w(0) = x$ and $(d/dt)\langle w(t), v\rangle|_{t=0} = \langle y, v\rangle$. Therefore w is a weak solution of (IV; x, y). To prove that w is the only weak solution of (IV; x, y), let $\overline{w}(t)$ be another weak solution of (IV; x, y) and set $u = w - \overline{w}$. Then $u(0) = 0$ and

$$(d/dt)\langle u(t), v\rangle = \int_0^t \langle u(s), A^*v\rangle ds$$

for all $v \in D(A^*)$ and $t \in [0, T]$. Consequently

$$\langle u(t), v\rangle = \left\langle \int_0^t \left[\int_0^s u(\sigma)d\sigma\right] ds, A^*v \right\rangle$$

for all $v \in D(A^*)$ and $t \in [0, T]$. Putting $z(t) = \int_0^t \left[\int_0^s u(\sigma)d\sigma\right] ds$ for $t \in [0, T]$, $z(t) \in D(A)$ and $Az(t) = u(t)$ by Lemma 1, and hence $z''(t) = Az(t)$ for $t \in [0, T]$ and $z(0) = z'(0) = 0$. It follows from (vii) that $z(t) = 0$, i.e., $\overline{w}(t) = w(t)$ for all $t \in [0, T]$.

Suppose that A is such that (IV; x, y) has, for each pair $[x, y] \in X \times X$, a unique weak solution. Let $w(t; x)$ be the weak solution of (IV; $x, 0$). For $x \in X$ and $t \in R$, define $C(t)x$ by

$$\begin{cases} C(t)x = w(t; x) - w(t; 0) & \text{if } t \in [0, T], \\ C(nT+s)x = 2C(nT)C(s)x - C(nT-s)x & \text{if } s \in (0, T] \text{ and } n = 1, 2, \cdots, \\ C(t)x = C(-t)x & \text{if } t < 0. \end{cases}$$

Note that

$$(3) \qquad \int_0^t \left[\int_0^s C(\sigma)x\, d\sigma\right] ds \in D(A) \quad \text{and} \quad C(t)x - x = A\int_0^t \left[\int_0^s C(\sigma)x\, d\sigma\right] ds$$

for all $x \in X$ and $t \in [0, T]$. In fact, it follows from the definition of $C(t)$ that for every $v \in D(A^*)$, $(d^2/dt^2)\langle C(t)x, v\rangle = \langle C(t)x, A^*v\rangle$ for a.e.t $\in [0, T]$. By integrating this equation twice and then by using Lemma 1, we obtain (3). Now let us prove that $\{C(t); t \in R\}$ satisfies the hypothesis of Lemma 2. To prove the linearity of $C(t)$ for $t \in [0, T]$, let α, β be a scalars and let $x, y \in X$. Set

$$u(t) = \alpha w(t; x) + \beta w(t; y) + (1 - \alpha - \beta)w(t; 0) \qquad \text{for } t \in [0, T].$$

Then u is a weak solution of (IV; $\alpha x+\beta y, 0$). By the uniqueness of weak solutions, we have $u(t)=w(t;\alpha x+\beta y)$ and hence $C(t): X\to X$ is linear for every $t\in[0,T]$. Define a map $\theta: X\to C([0,T];X)$ by $\theta(x)=C(\cdot)x$ for $x\in X$. To prove that θ is closed, let $x_n\to x$ in X and $\theta(x_n)=C(\cdot)x_n\to\varphi$ in $C([0,T];X)$. Since $C(t)x_n=x_n+A\int_0^t\left[\int_0^s C(\sigma)x_n\,d\sigma\right]ds$ by (3), it follows from the closedness of A that

$$\int_0^t\left[\int_0^s \varphi(\sigma)d\sigma\right]ds\in D(A) \quad\text{and}\quad \varphi(t)=x+A\int_0^t\left[\int_0^s\varphi(\sigma)d\sigma\right]ds$$

for $t\in[0,T]$. Hence $\varphi(t)+w(t;0)$ is a weak solution of (IV; x, 0), and then $\varphi(t)+w(t;0)=w(t;x)$ for $t\in[0,T]$, i.e., $\varphi=C(\cdot)x$, by the uniqueness of weak solutions. By virtue of the closed graph theorem, θ is bounded and $\|C(t)x\|\le\|\theta\|\,\|x\|$ for every $x\in X$ and $t\in[0,T]$. Thus, by the definition of $C(t)$, $\{C(t); t\in R\}$ is a one-parameter family in $B(X)$ such that $C(t)x: R\to X$ is continuous for every $x\in X$. We next show that (a) is satisfied. To this end, let $x\in D(A)$. By (3), we have

$$C(t)x-x=A\int_0^t\left[\int_0^s C(\sigma)x\,d\sigma\right]ds, \tag{4}$$

$$C(t)Ax-Ax=A\int_0^t\left[\int_0^s C(\sigma)Ax\,d\sigma\right]ds \tag{5}$$

for $t\in[0,T]$. Consider the function

$$z(t)=\int_0^t\left[\int_0^s C(\sigma)Ax\,d\sigma\right]ds-A\int_0^t\left[\int_0^s C(\sigma)x\,d\sigma\right]ds \qquad \text{for } t\in[0,T].$$

Since $C(\cdot)x\in C([0,T];X)$, it follows from (4) that $z\in C([0,T];X)$. Let $v\in D(A^*)$. Then $\langle z(t),v\rangle=\left\langle\int_0^t\left[\int_0^s C(\sigma)Ax\,d\sigma\right]ds, v\right\rangle-\left\langle\int_0^t\left[\int_0^s C(\sigma)x\,d\sigma\right]ds, A^*v\right\rangle$ is twice continuously differentiable in $t\in[0,T]$, $z(0)=0$ and $(d/dt)\langle z(t),v\rangle|_{t=0}=0$; and $(d^2/dt^2)\langle z(t),v\rangle=\langle z(t),A^*v\rangle$ for all $t\in[0,T]$ by (4) and (5). By the uniqueness of weak solution of (IV; x, y), we see that $z(t)=0$ for all $t\in[0,T]$. Combining this with (4), we have

$$C(t)x=x+\int_0^t\left[\int_0^s C(\sigma)Ax\,d\sigma\right]ds \qquad \text{for } t\in[0,T]; \tag{6}$$

and hence $C(t)x\in D(A)$ for all $t\in[0,T]$, and then $C(t)x\in D(A)$ for all $t\in R$. Finally, to see that (b) is satisfied, let $x\in D(A)$. It follows from (6) and the definition of $C(t)$ that $C(t)x: R\to X$ is twice continuously differentiable, $C(0)x=x$, $C'(0)x=0$ and $C''(t)x=C(t)Ax$ for $t\in R$. Moreover, by (5) and (6), $C(t)Ax=AC(t)x$ for $t\in R$. Using Lemma 2, $\{C(t); t\in R\}$ is a cosine family on X and A is its infinitesimal generator. Q.E.D.

Remarks. Let $w(t)$ be a weak solution of (IV; x, y).

1) Set $E=\left\{t\in[0,T]; (d/dt)\int_0^t g(s)ds\neq g(t)\right\}$. Then E is a null set. Integrating (1) we have

$$(d/dt)\langle w(t), v\rangle - \langle y, v\rangle = \left\langle \int_0^t w(s)ds, A^*v \right\rangle + \left\langle \int_0^t g(s)ds, v \right\rangle$$

for all $v \in D(A^*)$ and $t \in [0, T]$. Hence for every $v \in D(A^*)$

$$(d^2/dt^2)\langle w(t), v\rangle = \langle w(t), A^*v\rangle + \langle g(t), v\rangle \qquad \text{for } t \in [0, T]\backslash E.$$

2) Suppose that $w(t)$ is twice weakly differentiable for a.e.t $\in [0, T]$. By 1), there exists a null set $\tilde{E}$ $(\subset [0, T])$ independent of v such that

$$\langle (w - d^2/dt^2)w(t), v\rangle = \langle w(t), A^*v\rangle + \langle g(t), v\rangle$$

for all $v \in D(A^*)$ and $t \in [0, T]\backslash \tilde{E}$. Now, using Lemma 1, we obtain

$$w(t) \in D(A) \quad \text{and} \quad (w - d^2/dt^2)w(t) = Aw(t) + g(t) \qquad \text{for a.e.t} \in [0, T].$$

Acknowledgement. The author wishes to express his gratitude to Prof. I. Miyadera for his kind guidance and valuable advices.

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