31. *On the Absolute Nörlund Summability of Orthogonal Series*

By Yasuo OKUYAMA
Faculty of Engineering, Shinshu University, Nagano

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1978)

§1. Let $\sum a_n$ be any given infinite series with s_n as its n-th partial sum. If $\{p_n\}$ is a sequence of constants, real or complex, and

$$P_n = p_0 + p_1 + \cdots + p_n\,;\; P_{-k} = p_{-k} = 0, \qquad \text{for } k \geqq 1,$$

then the Nörlund mean t_n of $\sum a_n$ is defined by

$$(1.1) \qquad t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} a_k, \qquad (P_n \neq 0).$$

If the series

$$(1.2) \qquad \sum_{n=1}^{\infty} |t_n - t_{n-1}|$$

converges, then the series $\sum a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$.

In the special cases in which $p_n = A_n^{\alpha-1} = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$ and $p_n = 1/(n+1)$, summability $|N, p_n|$ are the same as the summability $|C, \alpha|$ and the absolute harmonic summability, respectively.

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a, b). We suppose that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x).$$

By $E_n^{(2)}(f)$, we denote the best approximation to $f(x)$ in the metric of L^2 by means of polynomials $\sum_{k=0}^{n-1} a_k \varphi_k(x)$, i.e., $\{E_n^{(2)}(f)\}^2 = \sum_{k=n}^{\infty} |a_k|^2$. We write

$$(1.3) \qquad W_k = \frac{1}{k} \sum_{n=k}^{\infty} \frac{n^2 p_n^2 p_{n-k}^2}{P_n^4} \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}} \right)^2$$

and

$$\Delta \lambda_n = \lambda_n - \lambda_{n-1}.$$

A denotes a positive absolute constant that is not always the same.

§2. The purpose of this paper is to give a general theorem on the almost everywhere summability $|N, p_n|$ of orthogonal series and deduce several known and new results from the theorem by the similar method as that used by Ul'yanov [7].

Our theorem reads as follows:

Theorem 1. *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$*

is a non-increasing sequence and the series $\sum_{n=1}^{\infty} 1/n\Omega(n)$ *converges. Let* $\{p_n\}$ *be non-negative and non-increasing. If the series* $\sum_{n=1}^{\infty} |a_n|^2 \Omega(n) W_n$ *converges, then the orthogonal series* $\sum_{n=1}^{\infty} a_n \varphi_n(x)$ *is summable* $|N, p_n|$ *almost everywhere, where* W_k *is defined by* (1.3).

We shall require the following lemmas.

Lemma 1 [1]. *If* $\{t_n\}$ *is defined by* (1.1), *then*

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n} p_{n-k}\left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right) a_k.$$

Lemma 2 [6]. *If we put* $p_n = A_n^{\alpha-1}$ *or* $1/(n+1)$, *then we have*

$$W_k = \frac{1}{k} \sum_{n=k}^{\infty} \frac{n^2 p_n^2 p_{n-k}^2}{P_n^4} \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 = \begin{cases} O(1), & \text{for } 1/2 < \alpha \leqq 1 \\ O(\log k), & \text{for } \alpha = 1/2 \\ O(k^{1-2\alpha}), & \text{for } 0 < \alpha < 1/2 \\ \text{or} & \\ O(k(\log k)^{-2}), & \text{for } p_n = 1/(n+1). \end{cases}$$

Proof of Theorem 1. By Lemma 1 and Schwarz inequality, we have

$$\begin{aligned} \int_a^b |\Delta t_n(x)|\, dx &\leqq (b-a)^{1/2} \left(\int_a^b |\Delta t_n(x)|^2\, dx \right)^{1/2} \\ &\leqq A \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{k=1}^{n} p_{n-k}^2 \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 |a_k|^2 \right\}^{1/2} \\ &\leqq A \frac{p_n}{P_n^2} \left\{ \sum_{k=1}^{n} p_{n-k}^2 \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 |a_k|^2 \right\}^{1/2}. \end{aligned}$$

Hence we have by Schwarz inequality

$$\begin{aligned} \sum_{n=1}^{\infty} \int_a^b |\Delta t_n(x)|\, dx &\leqq \sum_{n=1}^{\infty} \left\{ \frac{p_n^2}{P_n^4} \sum_{k=1}^{n} P_{n-k}^2 \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 |a_k|^2 \right\}^{1/2} \\ &= A \sum_{n=1}^{\infty} \frac{1}{n^{1/2}\Omega(n)^{1/2}} \left\{ \frac{n\Omega(n) p_n^2}{P_n^4} \sum_{k=1}^{n} p_{n-k}^2 \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 |a_k|^2 \right\}^{1/2} \\ &\leqq A \left(\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)} \right)^{1/2} \left\{ \sum_{n=1}^{\infty} \frac{n\Omega(n) p_n^2}{P_n^4} \sum_{k=1}^{n} p_{n-k}^2 \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 |a_k|^2 \right\}^{1/2} \\ &\leqq A \left\{ \sum_{k=1}^{\infty} |a_k|^2 \sum_{n=k}^{\infty} \frac{n\Omega(n) p_n^2 p_{n-k}^2}{P_n^4} \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 \right\}^{1/2} \\ &\leqq A \left\{ \sum_{k=1}^{\infty} |a_k|^2 \frac{\Omega(k)}{k} \sum_{n=k}^{\infty} \frac{n^2 p_n^2 p_{n-k}^2}{P_n^4} \left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2 \right\}^{1/2} \\ &\leqq A \left\{ \sum_{k=1}^{\infty} |a_k|^2 \Omega(k) W_k \right\} < \infty \end{aligned}$$

by virtue of the hypotheses of theorem. Thus this completes the proof of our theorem (see [6]).

Now, we consider some applications of our theorem. If we put

$\Omega(n)=\log n\,(\log\log n)^{1+\varepsilon}$ $(\varepsilon>0)$ in Theorem 1 and use Lemma 2, we have the following theorems.

Theorem 2 [7]. *If* $1\geqq\alpha>1/2$ *and* $\sum_{n=n_0}^{\infty}|a_n|^2\log n\,(\log\log n)^{1+\varepsilon}$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

Theorem 3 [7]. *If* $\sum_{n=n_0}^{\infty}|a_n|^2(\log n)^2(\log\log n)^{1+\varepsilon}$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|C,1/2|$ *almost everywhere.*

Theorem 4 [7]. *If* $0<\alpha<1/2$ *and* $\sum_{n=n_0}^{\infty}|a_n|^2 n^{1-2\alpha}\log n(\log\log n)^{1+\varepsilon}$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

Theorem 5. *If* $\sum_{n=n_0}^{\infty}|a_n|^2 n(\log n)^{-1}(\log\log n)^{1+\varepsilon}$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|N,1/n+1|$ *almost everywhere.*

Next, we suppose that $\Omega(0)=0$ and $W_0=0$. Then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}|a_n|^2\,\Omega(n)W_n&=\sum_{n=1}^{\infty}|a_n|^2\sum_{k=1}^{n}\Delta(\Omega(k)W_k)\\ &=\sum_{k=1}^{\infty}\Delta(\Omega(k)W_k)\sum_{n=k}^{\infty}|a_n|^2 \qquad (2.1)\\ &=\sum_{k=1}^{\infty}\Delta(\Omega(k)W_k)\{E_k^{(2)}(f)\}^2.\end{aligned}$$

By Lemma 2, we have

$$\Delta(\Omega(k)W_k)=\begin{cases}O(k^{-1}(\log\log k)^{1+\varepsilon}), & \text{for } 1/2<\alpha\leqq 1,\\ O(k^{-1}\log k(\log\log k)^{1+\varepsilon}), & \text{for } \alpha=1/2,\\ O(k^{-2\alpha}\log k(\log\log k)^{1+\varepsilon}), & \text{for } 0<\alpha<1/2,\\ O((\log k)^{-1}(\log\log k)^{1+\varepsilon}), & \text{for } p_n=1/(n+1).\end{cases} \qquad (2.2)$$

Hence, by (2.1) and (2.2), we can restate these theorems in the following forms, respectively.

Theorem 6 [7]. *If* $1\geqq\alpha>1/2$ *and* $\sum_{n=n_0}^{\infty}n^{-1}(\log\log n)^{1+\varepsilon}\{E_n^{(2)}(f)\}^2$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

Theorem 7 [7]. *If* $\sum_{n=n_0}^{\infty}n^{-1}\log n(\log\log n)^{1+\varepsilon}\{E_n^{(2)}(f)\}^2$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|C,1/2|$ *almost everywhere.*

Theorem 8 [7]. *If* $0<\alpha<1/2$ *and* $\sum_{n=n_0}^{\infty}n^{-2\alpha}\log n(\log\log n)^{1+\varepsilon}\{E_n^{(2)}(f)\}^2$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

Theorem 9. *If* $\sum_{n=n_0}^{\infty}(\log n)^{-1}(\log\log n)^{1+\epsilon}\{E_n^{(2)}(f)\}^2$ *converges, then the series* $\sum_{n=1}^{\infty}a_n\varphi_n(x)$ *is summable* $|N, 1/n+1|$ *almost everywhere.*

§ 3. Let $f(x)\in L^2(0,2\pi)$ and

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=0}^{\infty}A_n(x). \tag{3.1}$$

Let $\Omega(\delta,f)$ denote one of the following integral moduli:

$$\omega^{(2)}(\delta,f)=\sup_{0\leqq t\leqq\delta}\left\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2dx\right\}^{1/2},$$

$$\omega_2^{(2)}(\delta,f)=\sup_{0\leqq t\leqq\delta}\left\{\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2dx\right\}^{1/2},$$

$$w^{(2)}(\delta,f)=\left\{\frac{1}{\delta}\int_0^{\delta}\left(\int_0^{2\pi}[f(x+t)-f(x-t)]^2dx\right)dt\right\}^{1/2},$$

$$w_2^{(2)}(\delta,f)=\left\{\frac{1}{\delta}\int_0^{\delta}\left(\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2dx\right)dt\right\}^{1/2}.$$

Leindler [4] established the following equivalence theorem for the trigonometric system.

Theorem A. *Let* $0<\beta\leqq 2$. *Let* $\lambda(x)\,(x\geqq 1)$ *be a positive monotone function such that*

$$\sum_{k=n}^{\infty}\frac{1}{k^{\beta}\lambda(k)}\leqq A\frac{1}{n^{\beta-1}\lambda(n)}.$$

Then four conditions

$$\int_0^1\frac{1}{t^2\lambda(1/t)}\left(\int_0^{2\pi}[f(x+t)-f(x-t)]^2dx\right)^{\beta/2}dt<\infty,$$

$$\int_0^1\frac{1}{t^2\lambda(1/t)}\left(\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2dx\right)^{\beta/2}dt<\infty,$$

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}\Omega\left(\frac{1}{n},f\right)^{\beta}<\infty$$

and

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}\{E_n^{(2)}(f)\}^{\beta}<\infty$$

are mutually equivalent.

By Theorem A, we can obtain Theorems 10, 11, 12 and 13 from Theorems 6, 7, 8 and 9, respectively.

Theorem 10 [7]. *If* $1/2<\alpha\leqq 1$ *and*

$$\omega^{(2)}(\delta,f)=O\left((\log 1/\delta)^{-1/2}(\log\log 1/\delta)^{-1-\epsilon}\right),$$

then the Fourier series $\sum_{n=0}^{\infty}A_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

Theorem 11 [7]. *If* $\omega^{(2)}(\delta,f)=O\left((\log 1/\delta)^{-1}(\log\log 1/\delta)^{-1-\epsilon}\right)$, *then the Fourier series* $\sum_{n=0}^{\infty}A_n(x)$ *is summable* $|C, 1/2|$ *almost everywhere.*

Theorem 12 [7]. *If* $0<\alpha<1/2$ *and*

$$\omega^{(2)}(\delta, f)=O\,(\delta^{1/2-\alpha}(\log 1/\delta)^{-1}(\log\log 1/\delta)^{-1-\epsilon}),$$

then the Fourier series $\sum_{n=0}^{\infty} A_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

Theorem 13. *If* $\omega^{(2)}(\delta, f)=O\,(\delta^{1/2}(\log\log 1/\delta)^{-1-\epsilon})$, *then the Fourier series* $\sum_{n=0}^{\infty} A_n(x)$ *is summable* $|N, 1/n+1|$ *almost everywhere.*

We point out that both Theorems 12 and 13 can be also deduced from the theorem due to Lal [2, 3], who, however, stated nothing about the facts in the cited papers, but that neither Theorem 10 nor 11 can be induced from his theorem.

§4. Ul'yanov [7] showed that one cannot suppress the number $\varepsilon>0$ in Theorems 10, 11 and 12. In this section, we shall show that the number $\varepsilon>0$ is indispensable in Theorem 13.

The following theorem is due to Tsuchikura and Okuyama [6].

Theorem B. *Let* $\{p_n\}$ *be a positive non-increasing sequence such that for an integer* k_0, $p_{n-k}\left(\frac{P_n}{p_n}-\frac{P_{n-k}}{p_{n-k}}\right)=O(1)$ *for* $n>k_0\geqq k\geqq 1$.

If the series

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{k=1}^{n} p_{n-k}^2\left(\frac{P_n}{p_n}-\frac{P_{n-k}}{p_{n-k}}\right)^2(|a_k|^2+|b_k|^2)\right\}^{1/2} \tag{4.1}$$

converges, then almost all series of

$$\sum_{n=1}^{\infty}\pm(a_n\cos nx+b_n\sin nx) \tag{4.2}$$

are summable $|N, p_n|$ *for almost every* x, *and if the series* (4.1) *diverges, then almost all series* (4.2) *are non-summable* $|N, p_n|$ *for almost every* x *on a set of positive measure.*

Using this theorem, we can prove the following theorem.

Theorem 14. *There exists a function* $g(x)$ *belonging to* $L^2(0, 2\pi)$ *such that*

$$g(x)\sim\sum_{n=1}^{\infty} c_n\cos nx, \tag{4.3}$$

$$\omega^{(2)}(1/n, g)=O\,(n^{-1/2}(\log\log n)^{-1}) \tag{4.4}$$

and the series (4.3) *is non-summable* $|N, 1/n+1|$ *for almost every* x *on a set of positive measure.*

Proof. We put $p_n=1/(n+1)$ and

$$a_n=1/n\log\log n \qquad (n=1, 2, \cdots)$$

where we understand a_n to be zero if the right side is negative or lose its sense. Then there exists a function $f_0(x)$ belonging to $L^2(0, 2\pi)$ such that

$$f_0(x)\sim\sum_{n=1}^{\infty}\pm a_n\cos nx.$$

For this function $f_0(x)$, we have

$$E_n^{(2)}(f_0)=\left\{\sum_{k=n}^{\infty}\frac{1}{k^2(\log\log k)^2}\right\}^{1/2}$$
$$=O\left(\frac{1}{n^{1/2}\log\log n}\right).$$

Therefore, by a theorem of A. F. Timan and M. F. Timan (see [5], 331), we obtain

$$\omega^{(2)}\left(\frac{1}{n},f_0\right)\leqq\frac{A}{n}\sum_{\nu=0}^{n}E_\nu^{(2)}(f_0)=O\left\{\frac{1}{n^{1/2}\log\log n}\right\}.$$

On the other hand, if we put $b_n=0$ $(n=1,2,\cdots)$, we have

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{k=1}^{n}p_{n-k}^2\left(\frac{P_n}{p_n}-\frac{P_{n-k}}{p_{n-k}}\right)^2(|a_k|^2+|b_k|^2)\right\}^{1/2}$$
$$\geqq A\sum_{n=1}^{\infty}\frac{1}{n(\log n)^2}\left\{\sum_{k=[n/2]}^{n}\frac{k^2(\log n)^2}{(n-k+1)^2}\frac{1}{k^2(\log\log k)^2}\right\}^{1/2}$$
$$\geqq A\sum_{n=1}^{\infty}\frac{1}{n\log n\log\log n}\left\{\sum_{k=1}^{[n/2]}\frac{1}{k^2}\right\}^{1/2}$$
$$\geqq A\sum_{n=1}^{\infty}\frac{1}{n\log n\log\log n}=\infty.$$

Hence, by Theorem B with a suitable choice of a sequence of signs, putting

$$c_n=\pm a_n \qquad (n=1,2,\cdots),$$

we can conclude the existence of the required function $g(x)$.

References

[1] J. Banerji: On the absolute Nörlund summability factors (preprint).

[2] S. N. Lal: On the absolute Nörlund summability of Fourier series. Indian J. Math., **9**, 151–161 (1967).

[3] ——: Addendum to on the absolute Nörlund summability of Fourier series. Ibid., **10**, 167–168 (1968).

[4] L. Leindler: Über Strukturbedingungen für Fourierreihen. Math. Zeitschr., **88**, 418–431 (1965).

[5] A. F. Timan: Theory of Approximation of Functions of a Real Variable. Pergamon Press (1963).

[6] T. Tsuchikura and Y. Okuyama: On the absolute Nörlund summability factors of orthogonal series (to appear).

[7] P. L. Ul'yanov: Solved and unsolved problem in the theory of trigonometric and orthogonal series. Uspehi Math. Nauk., **19**, 3–69 (1964).