26. On the Bellman Transform of the Coefficients of Some Special Sine-series

By Takeshi KANO
Department of Mathematics, Okayama University, Okayama

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§ 1. Let $\{c_n\}$ be an infinite sequence of real numbers, and let $(Tc)_n$ denote the n-th arithmetic mean of $\{c_n\}$, i.e.

$$(Tc)_n = \frac{1}{n}\sum_{k=1}^{n} c_k.$$

It was Hardy [5] who proved that if

$$\sum_{n=1}^{\infty} c_n \sin nx \tag{1}$$

is the Fourier series of some L^p-function $f(x) \in L^p$, $p \geqq 1$, then

$$\sum_{n=1}^{\infty} (Tc)_n \sin nx \tag{2}$$

is the Fourier series of some L^p-function.

Bellman [2] introduced the transform

$$(T^*c)_n = \sum_{k=n}^{\infty} \frac{c_k}{k},$$

and proved that if (1) is the Fourier series of an $f(x) \in L^p$, $p>1$, then

$$\sum_{n=1}^{\infty} (T^*c)_n \sin nx \tag{3}$$

is the Fourier series of the class L^p. We note that we cannot here put $p=1$ in general, as is easily seen from the example

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\log^2 (n+1)}.$$

It seems still open to find the necessary and sufficient condition for (3) being the Fourier series of an L^1-function when (1) is the Fourier series of an $f(x) \in L^1$. The object of this note is to provide such necessary and sufficient conditions in the special case when $\{c_n\}$ is of bounded variation,[1] i.e.

$$\sum_{n=1}^{\infty} |\Delta c_n| < \infty, \tag{4}$$

where $\Delta c_n = c_n - c_{n+1}$.

We remark that for this special sequence $\{c_n\}$ G. and S. Goes [4] proved that a necessary and sufficient condition for (2) being the Fourier series of an L^1-function is

$$\sum_{n=1}^{\infty} \frac{|c_n|}{n} < \infty. \tag{5}$$

1) An infinite sequence of bounded variation converges to a finite limit.

An elementary proof of this fact, depending upon Theorem A below, is obtained by T. Kano [7].

§ 2. Our main result is the following

Theorem 1. *Let* $\{c_n\}$ *be of bounded variation and let* (1) *be a Fourier-Stieltjes series. Then* (3) *is the Fourier series of an* L^1*-function if and only if it is convergent in the metric of* L^1.

Before proving this theorem, we state below some theorems as requisites.

Theorem A ([10; Theorem 1], cf. [3; Theorem 5.4]). *Let a bounded real sequence* $\{c_n\}$ *be quasi-convex,*[2] *i.e.*

$$\sum_{n=1}^{\infty} n\,|\Delta^2 c_n| < \infty,$$

where $\Delta^2 c_n = \Delta c_n - \Delta c_{n+1}$. *Then* (1) *is a Fourier series if and only if* (5) *holds.*

Theorem B ([4; Theorem 5.3]). *Let* $\{c_n\}$ *be a real sequence of bounded variation. Then* (1) *is a Fourier-Stieltjes series if and only if it is a Fourier series.*

Theorem C (cf. [7; Corollary 1]). *If* $\{c_n\}$ *is of bounded variation, then* (5) *is necessary for* (1) *being a Fourier series.*

Theorem D ([8; Theorem 2]). *Let* $\{c_n\}$ *be a bounded, quasi-convex sequence of real numbers. Then the sine-series* (1) *converges in L if and only if* (5) *holds and*

$$|c_n| \log n \to 0 \quad (n \to \infty).$$

Proof of Theorem 1. It will suffice to prove the 'only if' part alone, since the 'if' part is known (cf. e.g. [1; Vol. I, Chap. I, § 12]). So we suppose that (3) is the Fourier series of an L^1-function. By Theorem B we know that (1) is indeed a Fourier series with $c_n \to 0$, and moreover (5) holds by Theorem C. On the other hand, a simple calculation shows that

$$\Delta(T^*c)_n = \frac{c_n}{n}, \qquad n\Delta^2(T^*c)_n = \Delta c_n + \frac{c_{n+1}}{n+1},$$

hence we have

$$\left| \sum_{n=1}^{\infty} n\,|\Delta^2(T^*c)_n| - \sum_{n=1}^{\infty} \frac{|c_{n+1}|}{n+1} \right| \leqq \sum_{n=1}^{\infty} |\Delta c_n|. \tag{6}$$

Thus $\{(T^*c)_n\}$ is quasi-convex from (5) and (6). Hence, by Theorem A, (3) is a Fourier series if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} |(T^*c)_n| < \infty, \tag{7}$$

i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{k=n}^{\infty} \frac{c_k}{k} \right| < \infty.$$

2) A bounded, quasi-convex sequence is of bounded variation.

Therefore, due to the absolute convergence, it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\frac{c_k}{k}=\sum_{k=1}^{\infty}\frac{c_k}{k}\sum_{n=1}^{k}\frac{1}{n}=\sum_{k=1}^{\infty}\frac{c_k}{k}\left(\log k+\gamma+O\left(\frac{1}{k}\right)\right)^{3)}$$

$$=\sum_{k=1}^{\infty}\frac{\log k}{k}c_k+\gamma\sum_{k=1}^{\infty}\frac{c_k}{k}+O\left(\sum_{k=1}^{\infty}\frac{|c_k|}{k^2}\right)$$

$$=\sum_{k=1}^{\infty}\frac{\log k}{k}c_k+O(1).$$

Consequently, if (3) is a Fourier series, then

$$(8)\qquad \sum_{k=1}^{\infty}\frac{\log k}{k}c_k$$

is convergent, which implies that

$$(9)\qquad (T^*c)_n=o\left(\frac{1}{\log n}\right).$$

But by Theorem D, (9) combined with (7) implies that (3) is convergent in L. Thus our proof is complete.

If in particular $\{c_n\}$ is nonnegative and nonincreasing, then we can obtain another necessary and sufficient condition.

Theorem 2. *When* $c_n\downarrow$ *and* (1) *is a Fourier-Stieltjes series of a function* $g(x)$ *of bounded variation,* (3) *is a Fourier series if and only if*

$$(10)\qquad g(x)\cdot\log^+\frac{1}{|x|}\in L.$$

Proof. The 'if' part is known to hold true in general from a theorem of Loo ([9; Theorem 5]). The 'only if' part follows from a theorem of Edmonds (cf. [3; Theorem 8.4]) that will assure (10) if (8) is convergent, because the function represented by the series

$$B+\sum_{n=1}^{\infty}\frac{\log n}{n}\sin nx$$

is positive on $[0,\pi]$ for some positive constant B, and it behaves just like the function $\log^+\frac{1}{|x|}$ (cf. e.g. [1; Vol. II, Chap. X, § 7], and T. Kano [6]).

References

[1] N. K. Bary: A Treatise on Trigonometric Series, vols. I and II. Pergamon Press (1964).

[2] R. Bellamn: A note on a theorem of Hardy on Fourier constants. Bull. Amer. Math. Soc., **50**, 741–744 (1944).

[3] R. P. Boas, Jr.: Integrability Theorem for Trigonometric Transforms. Springer (1967).

[4] G. and S. Goes: Sequences of bounded variation and sequences of Fourier coefficients. I. Math. Z., **118**, 93–102 (1970).

3) γ signifies as usual Euler's constant.

[5] G. H. Hardy: Notes on some points in the integral calculus. Messenger of Math., **58**, 50–52 (1929).
[6] T. Kano: Note on the special sine series. J. Fac. Sci. Shinshu Univ., **7**, 1–3 (1972).
[7] ——: Elementary proofs of some theorems on special Fourier series. Math. J. Okayama Univ., **19**, 19–23 (1976).
[8] T. Kano and S. Uchiyama: On the convergence in L of some special Fourier series. Proc. Japan Acad., **53A**, 72–77 (1977).
[9] Chin-Tsün Loo: Note on the properties of Fourier coefficients. Amer. J. Math., **71**, 269–282 (1949).
[10] S. A. Teljakovskiĭ: Some estimates for trigonometric series with quasi-convex coefficients (in Russian). Mat. Sbornik, **63** (105), 426–444 (1964); A. M. S. Translations (2) **86**, 109–131 (1970).