21. *On the Convergence in L of Some Special Fourier Series*

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It is well known that if a real function $f=f(x)$ belongs to the class $L^p=L^p(0,2\pi)$, $p>1$, then so does also its conjugate function $\tilde{f}$, and both of their Fourier series $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$ converge in (the metric of) L^p. Similarly, if $f\in L=L^1$ and if $0<p<1$, then we have $f\in L^p$ and $\tilde{f}\in L^p$, and their Fourier series $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$ again converge in L^p, though in this case the class L^p is not a metric space. When $p=1$, however, the situation becomes different, as was first noticed by F. Riesz (cf. e.g. [1-II; Chap. VIII, § 22]) who gave a counterexample showing that in the metric space L we cannot expect a corresponding result to hold true.[1] Actually, he constructed a function $f\in L$ and its conjugate $\tilde{f}\in L$ such that both $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$ unboundedly diverge in L, that is,

$$\limsup_{n\to\infty}\|S_n\|=\limsup_{n\to\infty}\|\tilde{S}_n\|=+\infty$$

which implies that

$$\limsup_{n\to\infty}\|f-S_n\|=\limsup_{n\to\infty}\|\tilde{f}-\tilde{S}_n\|=+\infty,$$

where $S_n=S_n(x)$ and $\tilde{S}_n=\tilde{S}_n(x)$ denote the n-th partial sums of $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$, respectively, and where $\|\ \|$ designates the ordinary L^1-norm (over the interval $(0,2\pi)$).

In the present article we shall be concerned with the problem of the convergence and divergence in L of Fourier series of sines and of cosines, with quasi-convex coefficients. Our primary aim is to prove Theorem 1 below, the proof itself of which is entirely of elementary and constructive character.

1. Theorems. Let (c_n) be an infinite sequence of real numbers. The sequence (c_n) is said to be of bounded variation, if it satisfies the condition $\sum_{n=1}^{\infty}|\varDelta c_n|<+\infty$, where $\varDelta c_n=c_n-c_{n+1}$, and (c_n) is quasi-convex, if $\sum_{n=1}^{\infty}n|\varDelta^2 c_n|<+\infty$, where $\varDelta^2 c_n=\varDelta c_n-\varDelta c_{n+1}$; a bounded, quasi-convex sequence (c_n) is of bounded variation.

Theorem 1. *We can find an infinite, quasi-convex null sequence (c_n) of non-negative real numbers such that the series*

$$\sum_{n=1}^{\infty}c_n\sin nx \tag{1}$$

1) This notwithstanding, it is true that if a trigonometric series converges in L to a function $f\in L$, then it is the Fourier series of the function f (cf. [1–I; Chap. I, §12]).

and

$$(2) \qquad \sum_{n=1}^{\infty} c_n \cos nx$$

are the Fourier series of a function $f \in L$ and its conjugate $\tilde{f} \in L$ respectively, but they do diverge in L boundedly, or unboundedly as well.

Our proof of this theorem is based upon the following Theorems 2 and 3 which may deserve an independent interest.

Theorem 2. *Let (c_n) be a bounded, quasi-convex sequence of real numbers. Then the sine series* (1) *converges in L if and only if*

$$(3) \qquad \sum_{n=1}^{\infty} \frac{|c_n|}{n} < +\infty$$

and

$$(4) \qquad |c_n| \log n \to 0 \qquad as\ n \to \infty.$$

Theorem 3. *Let the sequence (c_n) be as in Theorem* 2, *and let $S_n = S_n(x)$ denote the n-th partial sum of the sine series* (1). *Then we have*

$$(5) \qquad \limsup_{n\to\infty} \|S_n\| < +\infty$$

if and only if there hold (3) *and in addition*

$$(6) \qquad \limsup_{n\to\infty} |c_n| \log n < +\infty.$$

Proofs of Theorems 2 and 3 depend essentially upon the following two Theorems T and G.

Theorem T [7; Theorem 1]**.** *Let (c_n) be a quasi-convex null sequence of real numbers. Then the series* (1) *is the Fourier series of a function $f \in L$ if and only if the condition* (3) *is satisfied.*

Theorem G [3; Theorem 5.3]**.** *Let (c_n) be a real sequence of bounded variation. Then the series* (1) *is a Fourier-Stieltjes series if and only if it is a Fourier series. The same result holds true also for the series* (2), *if the sequence (c_n) is further assumed to be a null sequence.*

Also, we shall make use of the following well-known theorem due to W. H. Young and A. N. Kolmogorov (cf. [1–I; Chap. I, § 30], [1–II; Chap. X, § 2], and [8; Chap. V, Theorem 1.12]).

Theorem YK. *Let (c_n) be a quasi-convex null sequence of real numbers. Then the series* (2) *is the Fourier series of a function $\tilde{f} \in L$. It converges in L to $\tilde{f}$ if and only if the condition* (4) *is fulfilled. The L^1-norm of the n-th partial sum $\tilde{S}_n(x)$ of* (2) *is bounded if and only if the inequality* (6) *holds.*

We are now in a position to prove Theorems 2 and 3.

Proof of Theorem 2. The 'if' part: Suppose that the conditions (3) and (4) are satisfied. By Theorem T, the condition (3) together with the fact that $c_n \to 0$ as $n \to \infty$, which is a consequence of (4), implies that the series (1) is the Fourier series of an $f \in L$. On the other hand, owing to the quasi-convexity of (c_n), the series (2) is the Fourier series of an $\tilde{f} \in L$ and it converges in L by Theorem YK, on account of (4). Hence the series (1) also converges in L (cf. [1–II; Chap. VIII, § 22]).

The 'only if' part: Suppose that the series (1) converges in L. Then (1) is the Fourier series of an $f \in L$. Hence, again by Theorem T, the condition (3) is satisfied. On the other hand, since (c_n) is by assumption a quasi-convex sequence, (2) is the Fourier series of an $\tilde{f} \in L$, whence follows the convergence in L of the series (2). This implies (4).

Proof of Theorem 3. We shall give a proof of the 'only if' part alone, as the proof of the 'if' part is entirely parallel to that part of the proof of Theorem 2. So suppose that the inequality (5) holds. First we observe that (5) implies that (1) is a Fourier-Stieltjes series (cf. [1–I; Chap. I, § 60]). Then, by Theorem G, (1) is in fact a Fourier series, and it follows from this fact that (c_n) is a null sequence. Thus, (3) must hold by Theorem T. That (6) also holds will easily be seen just as in the proof of the 'only if' part of Theorem 2 above.

2. Constructing sequences. We are now going to prove Theorem 1. In view of Theorems 2 and 3, and of Theorem YK as well, it will obviously suffice to construct a quasi-convex null sequence (c_n) of non-negative real numbers that satisfies the condition (3) and either of the conditions

$$\limsup_{n\to\infty} c_n \log n = 1 \tag{7}$$

and

$$\limsup_{n\to\infty} c_n \log n = +\infty. \tag{8}$$

Existence of such a sequence (c_n) with (8) will show that, for a null sequence (c_n), the quasi-convexity of (c_n) and the condition (3) do not necessarily imply the condition (6) or, *a fortiori*, the condition (7).

Now, taking real numbers α, β and γ with $\alpha>0$ and $\beta>\gamma>1$, we set

$$n_m=[\exp (m \log^\alpha m)],\ k_m=[n_m/\log^{\beta-\gamma} m] \qquad (m>1),$$

where $[t]$ denotes the greatest integer not exceeding the real number t; we have then

$$k_m>0,\ n_m>2k_m,\ n_{m+1}-n_m>k_m+k_{m+1}+2 \qquad \text{for } m \geqq m_0,$$

provided $m_0=m_0(\alpha, \beta, \gamma)>2$ is chosen large enough. Define

$$c_n=\frac{1}{m \log^\beta m}-\frac{|k|}{n_m\cdot m \log^\gamma m}$$
$$\text{for } n=n_m+k,\ |k|\leqq k_m \quad (m\geqq m_0)$$

and

$$c_n=0 \qquad \text{for all other } n\geqq 1.$$

It is clear that $c_n\geqq 0$ for all n, $c_n\to 0$ as $n\to\infty$, and $c_n<1/(n_m\cdot\log^\gamma m)$ for $n=n_m\pm k_m$.

We have $\Delta^2 c_n=0$ unless $n=n_m-k_m-2,\ n_m-k_m-1,\ n_m-1,\ n_m+k_m-1$ or n_m+k_m for some $m\geqq m_0$, and for these exceptional values of n

$$|\Delta^2 c_n| \leqq A \frac{1}{n_m \cdot m \log^\gamma m} \qquad \text{with } A=1 \text{ or } 2.$$

Hence

$$\sum_n n|\Delta^2 c_n| \leqq \sum_{m \geqq m_0} \frac{1}{n_m \cdot m \log^\gamma m}\{(n_m-k_m-2)+2(n_m-k_m-1)$$
$$+2(n_m-1)+2(n_m+k_m-1)+(n_n+k_m)\}$$
$$<8 \sum_{m \geqq m_0} \frac{1}{m \log^\gamma m} < +\infty.$$

This proves the quasi-convexity of the sequence (c_n) defined above.

Next, we have

$$\sum_n \frac{|c_n|}{n} \leqq \sum_{m \geqq m_0} \frac{2k_m+1}{n_m-k_m} \frac{1}{m \log^\beta m} < 3 \sum_{m \geqq m_0} \frac{1}{m \log^\beta m} < +\infty,$$

which confirms (3).

Finally, we have for $m \geqq m_0$

$$\sup_{n \geqq n_m} c_n \log n \geqq \frac{1}{m \log^\beta m} \log n_m > \frac{1}{2} \log^{\alpha-\beta} m$$

and so, if we have chosen $\alpha > \beta$ then

$$\limsup_n c_n \log n = +\infty$$

which is (8). One may easily verify that, in order to realize (7), it is simply enough to take $\alpha=\beta$ *ab ovo*.

This completes the proof of Theorem 1.

3. Remarks. It might be remarked that our Theorems 2 and 3 were in a sense well-presented ones, which could be seen from the observations that follow. As a matter of fact, the following result is classical and is well known (cf. [2; § 5, p. 26]).

Theorem A. *If (c_n) is an infinite sequence of real numbers such that*

$$(9) \qquad \sum_{n=1}^{\infty} |\Delta c_n| \log (n+1) < +\infty \textit{ and } c_n \to 0 \qquad \textit{as } n \to \infty,$$

then both of (1) *and* (2) *converge in L, and hence they are the Fourier series of some functions belonging to L.*

On the other hand, it has been stated without proof by S. Szidon [6] (see also [5; Chap. VIII, § 403]) that one has

Theorem Sz. *If (c_n) is a real sequence satisfying*

$$(10) \qquad \sum_{n=1}^{\infty} |\Delta(c_n \log (n+1))| < +\infty,$$

then the cosine series (2) *is a Fourier series.*

Here, G. Goes [4] has shown, among other things, that the condition (9) is equivalent to

$$\sum_{n=1}^{\infty} |\Delta(c_n \log (n+1))| < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|c_n|}{n} < +\infty.$$

Thus, the condition (10) is truly weaker than (9). We note that, in the case of sine series (1), (10) ceases to be a sufficient condition for (1) being a Fourier series, as is readily seen from the well-known example with $(c_n)=(1/\log(n+1))$.

It is known that the condition (9) implies both of (3) and (4) (cf. [2; Theorem 5.1]). Therefore, in order to show that our Theorem 2, the 'if' part thereof, is indeed not contained in Theorem A, we must prove that the converse: "the condition (3), together with (4), implies (9)", does not hold[2] in general. We shall demonstrate this by an example.

If we take $\beta>\alpha>0$, $\beta>\gamma>1$, in the definition of the sequence (c_n) in § 2, then we get again a quasi-convex null sequence (c_n) of non-negative real numbers for which not only the condition (3) but also the conditions

$$c_n \log n \to 0 \quad \text{as } n\to\infty \tag{11}$$

and

$$\sum_{n=1}^{\infty} |\Delta(c_n \log(n+1))| = +\infty \tag{12}$$

are satisfied; here, as is easily seen, (12) is equivalent in this case to

$$\sum_{n=1}^{\infty} |\Delta c_n| \log(n+1) = \infty. \tag{13}$$

In fact, it will suffice to verify (11) and (12) for our (c_n). Now, (11) is obvious since $\alpha<\beta$, and we have for $m\geqq m_0=m_0(\alpha,\beta,\gamma)$

$$\sum_{\substack{n=n_m+k \\ |k|\leqq k_m}} |\Delta c_n| \log(n+1) = \frac{2}{\log^{\beta-\alpha} m} + O\left(\frac{1}{m \log^{\beta} m}\right);$$

on summing this up over $m\geqq m_0$, we obtain (13) and hence (12). We have thus proved that, for a quasi-convex sequence (c_n) of real numbers, the conditions (3) and (4) do not necessarily imply the condition (9).

References

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2) We note that R. P. Boas, Jr. [2; §5, p. 27, line 2] wrote the contrary; however, this might be a minor slip.

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