19. *Tail Probabilities for Positive Random Variables Satisfying Some Moment Conditions*

By Norio KÔNO
Institute of Mathematics, Yoshida College, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1977)

1. Let X be a positive random variable such that the asymptotic inequality

$$(c(1-\varepsilon))^{2n}\Gamma(2n+1)^{\beta}\leq E[X^{2n}]\leq(d(1+\varepsilon))^{2n}\Gamma(2n+1)^{\beta}$$

(n: integer)

holds for all ε, $0<\varepsilon<1$, where $0<c\leq d<+\infty$ and $0<\beta<1$. Then L. Davies [1] has proved the following inequalities as a corollary of his theorem:

$$\begin{aligned}\beta d^{-1/\beta}&\leq\varliminf_{x\to+\infty}-\log P(X\geq x)/x^{1/\beta}\\&\leq\varlimsup_{x\to\infty}-\log P(X\geq x)/x^{1/\beta}\\&\leq\beta d^{-1/\beta}(r_u/r_l)^{1/\beta},\end{aligned}$$

where $0<r_l\leq 1\leq r_u<+\infty$ are the two positive roots of $f(y)=0$,

$$f(y)=\beta(c/d)^{1/\beta}y^{1/\beta}/(1-\beta)-y/(1-\beta)+1.$$

We will extend his result to a class of positive random variables satisfying some moment conditions which includes his result. For this aim, we shall define "nearly regularly varying function with index α" which is first introduced in [2].

2. Let $\sigma(x)$ be a positive measurable function defined on $[c_0+\infty)$, $(c_0>0)$. We say that $\sigma(x)$ is a "nearly regularly varying function with index α" if and only if there exist two positive constants $r_1\geq r_2$ and a slowly varying function $s(x)$ such that

$$r_2x^{\alpha}s(x)\leq\alpha(x)\leq r_1x^{\alpha}s(x).$$

In particular, we say that $\sigma(x)$ is a "nearly slowly varying function" if $\alpha=0$.

As is well known (for example see [3]) $s(x)$ is represented as follows:

$$s(x)=b(x)\exp\int_{\cdot}^{x}a(t)/t\,dt,$$

where $a(x)$ and $b(x)$ are measurable functions such that

$$\lim_{x\to\infty}b(x)=b>0\quad\text{and}\quad\lim_{x\to\infty}a(x)=0.$$

3. Theorem 1. *Let X be a positive random variable. Assume that there exist two positive constants c_1 and h, and also a non-decreasing nearly regularly varying function $\sigma(x)$ with index α. $0<\alpha<1$, defined on $[1/h,+\infty)$ such that*

$$E[X^{2n}]\leq c_1^{2n}\prod_{k=1}^{2n}\sigma(k/h)$$

holds for any $n\geq n_0$. *Then*

$$P(X\geq A\sigma(x))\leq\exp\{-(\alpha-\delta(1/h))g(hx)\}$$

holds for any $x\geq 2n_0/h$, *where*

$$b_1=\sup_{x\geq 1/h} r_1b(x),\qquad b_2=\inf_{x\geq 1/h} r_2b(x),$$

$$A=c_1b_1/b_2,\qquad \delta(x)=\sup_{t\geq x}|a(t)|$$

and

$$g(y)=y-(1/2)\log y-\log\sqrt{2\pi}-2-0(1/y).$$

Theorem 2. *In addition to the conditions of Theorem* 1, *we assume that there exists a constant* $c_2(\leq c_1)$ *such that*

$$E[X^{2n}]\geq c_2^{2n}\prod_{k=1}^{2n}\sigma(k/h)$$

holds for any $n\geq n_0$. *Then*

$$\varliminf_{x\to\infty}\log P(X>A\sigma(x/h))/x\geq-\kappa(\alpha+\delta(1/h))\qquad(\kappa>0)$$

holds uniformly in $h\leq 1$. *In particular, we can choose* $\kappa=1$ *if* $c_1=c_2$, *and* $b_1=b_2$.

4. First we prove Theorem 1. By Chebyshev's inequality it follows that

$$P(X\geq A\sigma(x_n))\leq E[X^{2n}]/(A\sigma(x_n))^{2n}$$

$$\leq b_2^{2n}\sigma(x_n)^{-2n}\prod_{k=1}^{2n}(k/h)^{\alpha}\bar{s}(k/h),$$

where

$$\bar{s}(x)=\exp\int_{\cdot}^{x}a(t)/t\,dt\quad\text{and}\quad x_n=2n/h,\ n\geq n_0.$$

With elementary calculus by making use of Stirling's formula for $n!$, we have

$$\prod_{k=1}^{2n}\bar{s}(k/h)=\exp\left\{2n\int_{\cdot}^{2n/h}a(t)/t\,dt-\sum_{k=1}^{2n-1}k\int_{k/h}^{(k+1)/h}a(t)/t\,dt\right\}$$

and

$$\sum_{k=1}^{2n-1}k\int_{k/h}^{(k+1)/h}|a(t)|/t\,dt\leq\delta(1/h)\sum_{k=1}^{2n-1}k\log(1+1/k)$$

$$=\delta(1/h)(2n-(1/2)\log 2n-\log\sqrt{2\pi}-0(1/n)).$$

Again by Stirling's formula, we have

$$b_2^{2n}\sigma(x_n)^{-2n}\prod_{k=1}^{2n}(k/h)^{\alpha}\bar{s}(k/h)$$

$$\leq\exp\{-(\alpha-\delta(1/h))(2n-(1/2)\log 2n-\log\sqrt{2\pi}-0(1/n))\}.$$

Finally, it follows for $x_n\leq x\leq x_{n+1}$ that

$$P(X\geq A\sigma(x))\leq P(X\geq A\sigma(x_n))$$

$$\leq\exp\{-(\alpha-\delta(1/h))(hx_n-(1/2)\log hx_n-\log\sqrt{2\pi}-0(1/n)\}$$

$$=\exp\{-(\alpha-\delta(1/h))g(hx)\}.$$

Now we prove Theorem 2. Setting

$$F(x)=P(X\leq x),\ B=(1+\varepsilon)A,\ m=[\kappa n]\ (\kappa>1),\quad\text{and}\quad p=[\mu m]\ (\mu>0),$$

we have

$$P(X>A\sigma(x_n))\geq\int_{A\sigma(x_n)+0}^{B\sigma(x_n)}dF(x)$$
$$\geq(B\sigma(x_n))^{-2m}\left(\int_0^{+\infty}-\int_0^{A\sigma(x_n)}-\int_{B\sigma(x_n)+0}^{+\infty}\right)x^{2m}dF(x).$$

Let us estimate each term.

$$\mathrm{I}=\int_0^{\infty}x^{2m}dF(x)=E[X^{2m}]$$
$$\geq(c_2b_2)^{2m}\exp\left\{2m\left(\alpha\log(2m/h)+\int_{\cdot}^{2m/h}a(t)/tdt\right)\right.$$
$$\left.-\alpha(2m-(1/2)\log 2m-\log\sqrt{2\pi}-0(1/m))-\sum_{k=1}^{2m-1}k\int_{k/h}^{(k+1)/h}a(t)/tdt\right\},$$
$$\mathrm{II}=\int_0^{A\sigma(x_n)}x^{2m}\,dF(x)=\left(\int_0^{A\sigma(x_{n_0})}+\int_{A\sigma(x_{n_0})+0}^{A\sigma(x_n)}\right)x^{2m}dF(x)=\mathrm{II}_1+\mathrm{II}_2,$$
$$\mathrm{II}_1\leq(A\sigma(x_{n_0}))^{2m},$$
$$\mathrm{II}_2\leq(A\sigma(x_{n_0+1}))^{2m}(1-F(A\sigma(x_{n_0})))+\sum_{k=n_0+1}^{n-1}(A\sigma(x_{k+1}))^{2m}(1-F(A\sigma(x_k)))$$
$$=\mathrm{II}_{2,1}+\mathrm{II}_{2,2},$$
$$\mathrm{II}_{2,1}\leq(Ab_1)^{2m}\exp\left\{2m\left(\alpha\log(2n_0+2)/h+\int_{\cdot}^{(2n_0+2)/h}a(t)/tdt\right)\right.$$
$$\left.-(\alpha-\delta(1/h))\left(2n_0-\frac{1}{2}\log 2n_0-\log\sqrt{2\pi}-O\left(\frac{1}{n_0}\right)\right)\right\},$$
$$\mathrm{II}_{2,2}\leq(A\sigma(x_n))^{2m}\exp\left\{-\alpha\left(2n-2-\frac{1}{2}\log(2n-2)-\log\sqrt{2\pi}-O\left(\frac{1}{n-1}\right)\right)\right.$$
$$\left.-\sum_{k=1}^{2n-3}k\int_{k/h}^{(k+1)/h}a(t)/tdt\right\}\sum_{k=n_0+1}^{n-1}(\sigma(x_{k+1})/\sigma(x_n))^{2m}$$
$$\times\exp\left\{\alpha\left(2n-2k+2-\frac{1}{2}\log(n-1)/k+O\left(\frac{1}{k}\right)-O\left(\frac{1}{n-1}\right)\right)\right.$$
$$\left.+\sum_{j=2k}^{2n-3}j\int_{j/h}^{(j+1)/h}a(t)/tdt\right\}$$
$$\leq(A\,b_1)^{2m}\exp\left\{2m\left(\alpha\log 2n/h+\int_{\cdot}^{2n/h}a(t)/tdt\right)\right.$$
$$-\alpha\left(2n-2-\frac{1}{2}\log(2n-2)-\log\sqrt{2\pi}-O\left(\frac{1}{n-1}\right)\right)$$
$$\left.-\sum_{j=1}^{2n-3}j\int_{j/h}^{(j+1)/h}a(t)/tdt\right\}\sum_{k=0}^{\infty}\exp\{-2k(\alpha(\kappa-1)-2\kappa\delta(2n_0/h))\},$$
$$\mathrm{III}=\int_{B\sigma(x_n)+0}^{+\infty}x^{2m}dF(x)\leq(B\sigma(x_n))^{-2p}E[X^{2(m+p)}]$$
$$\leq(1+\varepsilon)^{-2p}(c_1b_1)^{2m}\exp\left\{2m\left(\alpha\log(2m+2p)/h+\int_{\cdot}^{(2m+2p)/h}a(t)/tdt\right)\right.$$
$$+2p\left(\alpha\log(m+p)/n+\int_{2n/h}^{(2m+2p)/h}a(t)/tdt\right)$$
$$-\alpha\left(2m+2p-\frac{1}{2}\log(2m+2p)-\log\sqrt{2\pi}-O\left(\frac{1}{m+p}\right)\right)$$
$$\left.-\sum_{k=1}^{2m+2p-1}k\int_{k/h}^{(k+1)/h}a(t)/tdt\right\}.$$

Therefore, for suitably chosen κ and ε we have

$$\lim_{n\to+\infty} (\mathrm{II}+\mathrm{III})/\mathrm{I}=O$$

uniformly in $h\leq 1$. This yields the proof of Theorem 2.

References

[1] Davies, L.: Tail probabilities for positive random variables with entire characteristic functions of very regular growth. Z. Angew. Math. Mech., **56**(3), 334–336 (1976).

[2] Kôno, N.: On the modulus of continuity of sample functions of Gaussian processes. J. Math. Kyoto Univ., **10**(3), 493–536 (1970).

[3] Seneta, E.: Regularly varying functions. Lecture Notes in Mathematics, Springer-Verlag, 508 (1976).