3. *Studies on Holonomic Quantum Fields.* I

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To understanding the mathematical structure of quantized fields or systems with infinite freedom, non trivial but exactly calculable models would be of great help [1]. In this and subsequent notes we present, both in the continuum and in the lattice, 2-dimensional soluble models of neutral scalar massive field theory whose τ-functions exhibit a non trivial singularity structure.

In the present article we deal with the continuum case. We introduce an auxiliary free fermi/bose field and construct the field operator by specifying its induced rotation in the space of wave functions. Making use of the "theory of rotation" (2 cf. [2]) developed recently by the first author, we express this field operator in the normal product form of these free fields. We also calculate the asymptotic fields and the S-matrix of the field φ^F defined in 3. Next we give explicit formulae for τ-functions of these models and study their holonomy structure.

The lattice field theory will be discussed in a subsequent paper. Specifically we shall show that our model φ^F/φ_F coincide with the scaling limit of the Ising model from above/below the critical temperature. Main part of these results has been announced in [3].

We use the following notations. The space-time and the energy-momentum co-ordinates are denoted by $x=(x^0, x^1)$ and $p=(p^0, p^1)$. We also use $x^{\pm}=(x^0 \pm x^1)/2$ and $p^{\pm}=p^0 \pm p^1$. The mass-shell $\{p \in \boldsymbol{R}^2 \mid p^2=(p^0)^2-(p^1)^2=m^2\}$ $(m>0)$ is denoted by M. For $p \in M$ we set $u^{\pm 1}=p^{\pm}/m$, $\underline{du}=du/2\pi|u|$.

1. Let $\psi(u)^{\dagger}$ and $\psi(u)$ $(u>0)$ be the creation and annihilation operators of auxiliary fermion. If we define $\psi(u)=\psi(-u)^{\dagger}$ for $u<0$, their anti-commutation relation reads $[\psi(u), \psi(u')]_+=2\pi|u|\delta(u+u')$. Likewise we define auxiliary bosons $\phi(u)$ with the commutation relation $[\phi(u), \phi(u')]_-=2\pi u\delta(u+u')$. In two dimensional space-time these two are in fact equivalent. Namely

$$(1) \qquad \phi_{\pm}(u)=:\psi(u) \exp \int_0^{\infty} (-2)\theta(\pm(|u|-u'))\psi(u')^{\dagger}\psi(u')\underline{du'}:$$

satisfy the commutation relation $[\phi_{\pm}(u), \phi_{\pm}(u')]_-=2\pi u\delta(u+u')$, and conversely $\psi(u)$ is given by

$$(2) \qquad \psi(u)=:\phi_{\pm}(u) \exp \int_0^{\infty} (-2)\theta(\pm(|u|-u'))\phi_{\pm}(u')^{\dagger}\phi_{\pm}(u')\underline{du'}:.$$

2. We let W denote an orthogonal/symplectic space, a vector space equipped with a non-degenerate symmetric/skew-symmetric inner product $\langle w, w'\rangle$. First consider the orthogonal case and denote by $A(W)$ the enveloping algebra (Clifford algebra) over W with defining relation $[w, w']_+ = \langle w, w'\rangle$. $G(W)$ denotes the Clifford group $\{g \in A(W) \mid \exists g^{-1},\ gWg^{-1} = W\}$. Let $g \mapsto g^*$ denote the anti-automorphism of $A(W)$ characterized by $w^* = w$ for $w \in W$. Set $n(g) = g^*g = gg^*$ for $g \in G(W)$, and $g \mapsto n(g)$ will define a group homomorphism $G(W) \to GL(1)$. Let $W = V^\dagger \oplus V$ be a decomposition into two holonomic subspaces. This means that there exist a basis $\psi^\dagger = (\psi^\dagger_\mu)$ of $V^\dagger$ and a basis $\psi = (\psi_\mu)$ of V such that $\langle \psi^\dagger_\mu, \psi^\dagger_\nu\rangle = 0$, $\langle \psi_\mu, \psi_\nu\rangle = 0$ and $\langle \psi^\dagger_\mu, \psi_\nu\rangle = \delta_{\mu\nu}$. Then $A(W)$ is a semi-direct product of two exterior algebras $\Lambda(V^\dagger)$ and $\Lambda(V)$, and a $\Lambda(V^\dagger)$-$\Lambda(V)$-isomorphism $N: A(W) = \Lambda(V^\dagger) \cdot \Lambda(V) \to \Lambda(W) = \Lambda(V^\dagger) \wedge \Lambda(V)$ such that $N(1) = 1$ is determined uniquely. The image $N(g) \in \Lambda(W)$ we call the norm of g. (In physicists' notation $g = :N(g):$.) For $g \in G(W)$ $T_g: w \in W \mapsto gwg^{-1} \in W$ is a rotation, an isomorphism which preserves the inner product. Let $T_g(\psi^\dagger, \psi) = (\psi^\dagger, \psi)\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. First assume that T_4 is invertible. Then we have the following expression of the norm of g.

(3) $N(g) = \langle g\rangle \exp((1/2)\psi^\dagger T_2 T_4^{-1\,t}\psi^\dagger + \psi^\dagger({}^tT_4^{-1} - 1)^t\psi + (1/2)\psi T_4^{-1} T_3{}^t\psi)$,

where $n(g) = \langle g\rangle^2 (\det T_4)^{-1}$, and we regard $\psi^\dagger_\mu, \psi_\mu$ as elements of $\Lambda(W)$. Next we assume that $\dim \operatorname{Ker} T_4 = 1$, and choose $\psi^\dagger_0 \in V^\dagger$, $\psi_0 \in V$ and $w \in G(W) \cap W$ such that $T_g\psi_0 = \psi^\dagger_0$, $w^2 = 1$ and $\langle w, \psi^\dagger_0\rangle = 1$. Then $(T_{wg})_4$ is invertible and

(4) $$N(g) = \psi^\dagger_0 N(wg) + N(wg)\psi_0.$$

Here we regard $\psi^\dagger_0$ and ψ_0 as elements of $\Lambda(W)$.

Next consider the symplectic case, and define $A(W)$, $G(W)$, etc. with due modifications. In particular $w^* = iw$ for $w \in W$, and the norm of $g \in A(W)$ is defined as an element of the symmetric tensor algebra $S(W)$. Assuming that T_4 is invertible, we have

(5) $N(g) = \langle g\rangle \exp((1/2)\phi^\dagger(-T_2T_4^{-1})^t\phi^\dagger + \phi^\dagger({}^tT_4^{-1} - 1)^t\phi + (1/2)\phi T_4^{-1}T_3{}^t\phi)$,

with $n(g) = \langle g\rangle^2 \det T_4$.

3. Let now W be the space of wave functions $w(x) = (w_+(x), w_-(x))$ satisfying the Dirac equation $\partial w_\pm(x)/\partial x^\pm \mp m w_\mp(x) = 0$. An orthogonal structure is introduced to W by defining $\langle w, w'\rangle = \int_{-\infty}^{+\infty} dx^1 (w_+(x)w'_+(x) + w_-(x)w'_-(x))$. If we identify $w \in W$ with the operator $\int_{-\infty}^{+\infty} dx^1(w_+(x)\psi_+(x) + w_-(x)\psi_-(x))$, where $\psi_\pm(x) = \int_{-\infty}^{+\infty} \underline{du}\sqrt{0 + iu}^{\pm 1}\psi(u) \exp(-im(x^- u + x^+ u^{-1}))$, the Clifford algebra $A(W)$ is nothing but the operator algebra of free fermions. We choose as $V^\dagger/V$ the set of creation/annihilation operators in W. Set $W_x^{\pm}=\{w\in W\,|\,w(x')=0 \text{ if } (x'-x)^2<0,\ x'^1-x^1\lessgtr 0\}$, and we shall have $W=W_x^+\oplus W_x^-$, an orthogonal decomposition. We now introduce our field operator $\varphi_F(x)\in A(W)$ by specifying its induced rotation $T_{\varphi_F(x)}$ with the property $T_{\varphi_F(x)}{}^2=1$ by

$$(6)\qquad T_{\varphi_F(x)}(w^+ + w^-)=w^+ - w^-,\qquad w^{\pm}\in W_x^{\pm}.$$

Applying the formula (3) to the present situation and choosing $\langle\varphi_F(x)\rangle=1$ we obtain the following expression for $\varphi_F(x)$:

$$(7)\qquad \varphi_F(x)=\,:\exp L_F(x):\,,$$

where $L_F(x)=(1/2)\iint_{-\infty}^{+\infty}\underline{du}\,\underline{du'}\,\dfrac{-i(u-u')}{u+u'-i0}\psi(u)\psi(u')\exp(-im(x^-(u+u')+x^+(u^{-1}+u'^{-1})))$. The micro-causality and the Lorentz covariance of $\varphi_F(x)$ are manifest in this approach.

We construct $\varphi^F(x)$ and $\varphi_B(x)$ analogously, using the formulae in the case dim Ker $T_4=1$ and in the symplectic case, respectively. In the latter case we choose as W the solution space to the Klein-Gordon equation and equip it with the inner product $\langle w, w'\rangle=-i\int_{-\infty}^{+\infty}dx^1(w(x)\cdot\partial w'(x)/\partial x^0-\partial w(x)/\partial x^0\cdot w'(x))$. The results are

$$(8)\qquad \varphi^F(x)=\,:\psi_0(x)\exp L_F(x):\,,$$

where $\psi_0(x)=\int_{-\infty}^{+\infty}\underline{du}\,\psi(u)\exp(-im(x^-u+x^+u^{-1}))$,

$$(9)\qquad \varphi_B(x)=\,:\exp L_B(x):\,,$$

where
$$L_B(x)=(1/2)\iint_{-\infty}^{+\infty}\underline{du}\,\underline{du'}\,\frac{-2\sqrt{u-i0}\sqrt{u'-i0}}{u+u'-i0}\phi(u)\phi(u')\times\exp(-im(x^-(u+u')+x^+(u^{-1}+u'^{-1}))).$$

4. The asymptotic fields for φ^F are defined by

$$(10)\qquad \phi_{\pm}(x)=\int_{-\infty}^{+\infty}\underline{du}\,\phi_{\pm}(u)\exp(-ipx),$$

where $\phi_{\pm}(u)=\lim_{t\to\pm\infty} i\int_{x^0=t}dx^1(\varphi^F(x)(\partial/\partial x^0)\exp(ipx)-(\partial/\partial x^0)\varphi^F(x)\exp(ipx))$. We find that this limit coincides with $\phi_{\pm}(u)$ defined in 1. The asymptotic states $|\rangle_{\pm}$ are related to the auxiliary fermion states $|\rangle_F$ through the formulae

$$(11)\qquad |u_n,\cdots,u_1\rangle_{\pm}=\prod_{i<j}\varepsilon(\pm(u_i-u_j))\,|u_n,\cdots,u_1\rangle_F,$$

where $\varepsilon(u)$ stands for the signature of u. Accordingly the particle number is conserved, and the S-matrix in the n-particle state is given by $(-)^{n(n-1)/2}$ times the identity operator, showing that the maximum phase shift is attained in this model.

5. The n-point τ-function of an operator $\varphi(x)$ is expressed as follows:

$$(12)\qquad \tau_n(p_1,\cdots,p_n)=\sum_{\text{permutations}}^{n!}T_{n-1}(p_1,p_1+p_2,\cdots,p_1+\cdots+p_{n-1})\times(2\pi)^2\delta^2(p_1+\cdots+p_n),$$

where

$$T_{n-1}(q_1,\cdots,q_{n-1})=\sum_{\nu}(1/\nu_1!\cdots\nu_{n-1}!)\int_0^\infty\cdots\int_0^\infty \underline{du}$$
$$\times\prod_{j=1}^{n}\varphi_{\nu_j+\nu_{j-1}}(u_{j\nu_j},\cdots,u_{j1},-u_{j-1\,1},\cdots,-u_{j-1\,\nu_{j-1}})$$
$$\times\prod_{j=1}^{n-1}2\pi\delta(q_j^+-mU_j)i(q_j^--mU'_j+i0)^{-1},$$

with $U_j=\sum_{k=1}^{\nu_j}u_{jk}$, $U'_j=\sum_{k=1}^{\nu_j}u_{jk}^{-1}$ and $\nu_0=\nu_n=0$. The (anti-)symmetric functions φ_n are the matrix elements defined by $\varphi_n(u_1,\cdots,u_n)=\langle -u_{m+1},\cdots,-u_n|\varphi(0)|u_1,\cdots,u_m\rangle$ for $u_1,\cdots,u_m>0$ and $u_{m+1},\cdots,u_n<0$. In our models they are obtained from (7), (8) and (9).

(13) $$\varphi_{F,n}(u_1,\cdots,u_n)=\text{Pfaffian}\left(iP\frac{u_j-u_k}{u_j+u_k}\right)_{1\leq j,\,k\leq n}$$
$$=\begin{cases} i^{n/2}\prod_{1\leq j<k\leq n}P\dfrac{u_j-u_k}{u_j+u_k} & (n\text{ even})\\ 0 & (n\text{ odd}),\end{cases}$$

(14) $$\varphi_n^F(u_1,\cdots,u_n)=-i\varphi_{F,n+1}(\infty,u_1,\cdots,u_n)$$
$$=\begin{cases} 0 & (n\text{ even})\\ i^{(n-1)/2}\prod_{1\leq j<k\leq n}P\dfrac{u_j-u_k}{u_j+u_k} & (n\text{ odd}),\end{cases}$$

and

(15) $$\varphi_{B,n}(u_1,\cdots,u_n)=\text{Hafnian}\left(-2P\frac{\sqrt{u_j-i0}\sqrt{u_k-i0}}{u_j+u_k}\right)_{1\leq j,\,k\leq n}.$$

Here $P(1/(u+v))$ denotes the principal value of $1/(u+v)$, and for a symmetric matrix $(a_{jk})_{1\leq j,\,k\leq n}$ we set Hafnian$(a_{jk})=0$ for odd n and $=\sum' a_{j_1j_2}a_{j_3j_4}\cdots a_{j_{n-1}j_n}$ for even n, where the sum is taken over $(n-1)!!$ pairings of indices $1,\cdots,n$. In particular the (Euclidean) two point functions of φ_F and φ^F coincide with those obtained by [4] and [5].

The singularity/holonomy spectrum of $\tau_n(p)$ is confined to the union of positive-α/complex Landau singularities corresponding to graphs with no internal vertices [6], where the number of (internal and external) lines incident to each vertex is always even for φ^F and is always odd for φ_F, φ_B. On the leading singularity Λ_G^+, the order of τ_n for φ^F or φ_F is given by

(16) $$\text{ord}_{\Lambda_G^+}\tau_n=n_e-N/2-\sum_{i<j}N_{ij}(N_{ij}-1)/2,$$

where n_e denotes the number of vertices of G, N_{ij} the number of internal lines joining the vertices i and j, and $N=\sum_{i<j}N_{ij}$. Note that repulsive effect of multiple internal lines is incorporated in (16).

Finally we remark that the generalized unitarity relation for the τ-function of φ^F

$$0=\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{k=0}^{n}(-)^k\sum_{\text{combinations}}^{\binom{n}{k}}\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{l}\underline{du_i}\tau_k^{(l)}(p_1,\cdots,p_k;u_1,\cdots,u_l)$$

$$\times \overline{\bar{\tau}^{(l)}_{n-k}(-p_{k+1}, \cdots, -p_n; u_1, \cdots, u_l)}$$

where we set $\tau^{(l)}_k(p_1, \cdots, p_k; u_1, \cdots, u_l) = \tau_{k+l}(p_1, \cdots, p_k, q_1, \cdots, q_l) \times \prod_{i=1}^{l} (q_i^2 - m^2)|_{q_i^{\pm} \to u_i^{\pm 1}}$ and bar denotes the complex conjugation, is directly and analytically verified by using our explicit formulae (12) and (14).

References

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