16. *On the Periods of Enriques Surfaces. II*

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This is a continuation of [4], and here we report on our result on the image of the period map for Enriques surfaces.

Let S be an Enriques surface defined over the field of complex numbers. Then there corresponds to S a point $\lambda(S)$, called the period of S, which is in the period space D/Γ. First we recall the construction of D and Γ. Let T be the universal covering of S. Then T is a $K3$ surface, and hence the homology group $H_2(T, \boldsymbol{Z})$, given with the intersection product, is isomorphic to a unique even unimodular euclidean lattice Λ of signature (3, 19). Moreover, if we associate the involution τ induced by the covering transformation, the pair $(H_2(T, \boldsymbol{Z}), \tau)$ is isomorphic to a standard pair (Λ, ρ) (see [4], § 3). Let $\Lambda(-1)$ denote the (-1)-eigenspace of ρ. Then D consists of non-zero linear maps $\omega: \Lambda(-1) \to \boldsymbol{C}$, modulo multiplications by constants, which satisfy the Riemann bilinear relations

$$\omega \cdot \omega = 0, \qquad \omega \cdot \bar{\omega} > 0,$$

the product being induced by that on $\Lambda(-1)$. On the other hand, Γ is the group of those automorphisms of $\Lambda(-1)$ which are the restrictions of the automorphisms of Λ commuting with ρ.

An element e of $\Lambda(-1)$ is called a *root* if it satisfies $e^2 = -2$. From the explicit description of $\Lambda(-1)$ in [4], we infer that such elements exist. If e is a root, we define a hypersurface H_e of D by the condition $\omega(e) = 0$. We shall use H_e/Γ to denote $H_e\Gamma/\Gamma$.

Main Theorem. *There exists only a finite number of Γ-equivalence classes of the roots e in $\Lambda(-1)$, and if λ is a point of D/Γ outside of the union of the hypersurfaces H_e/Γ, then λ is the period of an Enriques surface S, which is uniquely determined by λ. Moreover, any point of H_e/Γ is not the period of an Enriques surface.*

The basic idea of the proof is that of [3].

First, by the construction in [4], each Enriques surface S is birationally equivalent to a double covering of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. We take a system of 2-way homogeneous coordinates $(Y_1, Y_2; Z_1, Z_2)$ and fix the projection onto the second factor. Then the branch locus of the covering consists of the two fibres Γ_i defined by $Z_i = 0$, $i = 1, 2$, and a curve B_E^0 of bidegree (4, 4), which has two 2-fold double points at P_i on Γ_i, having the contact of order 4 with Γ_i at P_i, $i = 1, 2$. An Enriques surface S, with an elliptic pencil being specified, is said to be *of special type*, if P_1 and P_2 are on a section $Y_1=\beta Y_2$ for some constant β (possibly ∞). Suppose this is not the case. Then we may assume that P_i is given by $Y_i=Z_i=0$, $i=1,2$. Hence B_E^0 is defined by a linear combination of the monomials

$$Y_1^i Y_2^{4-i} Z_1^j Z_2^{4-j}, \qquad 4 \leqq i+2j \leqq 8.$$

In order to obtain a model for the universal covering T, we consider the double covering $\pi: \boldsymbol{P}^1 \to \boldsymbol{P}^1$ branched at $Z_1=0$ and $Z_2=0$. Pulling B_E^0 back by π, and applying two elementary transformations, we see that T is birationally equivalent to a double covering of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, whose branch locus B is of bidegree $(4,4)$, and is defined by a linear combination of the monomials

$$(1) \qquad Y_1^i Y_2^{4-i} Z_1^j Z_2^{4-j}, \qquad i+j \equiv 0 \mod 2.$$

The covering transformation is induced by

$$(2) \qquad I: (Y_1, Y_2; Z_1, Z_2) \to (-Y_1, Y_2; -Z_1, Z_2),$$

and the interchange of the sheets of $T \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

Next we consider Baily-Borel's compactification $(D/\Gamma)^*$ of D/Γ. Suppose that the branch locus B degenerates into a singular one. Then we take a 1-parameter family of the divisors defined by (1), whose generic member is non-singular. This determines a point in $(D/\Gamma)^*$, which can be thought of as the period corresponding to B. Note that this point may depend on the choice of the 1-parameter family.

If B passes through a fixed point of I defined by (2), say $Y_1=Z_1=0$, then the corresponding double covering T is still birationally equivalent to a $K3$ surface. T has a double point over $Y_1=Z_1=0$, and this is a fixed point of the involution ι induced by I and the interchange of the sheets. Therefore the quotient space T/ι is not an Enriques surface, but a rational surface with a rational quadruple point. In this case the period is in H_e/Γ for some root e.

If B has infinitely near triple points, a quadruple point, or a double component, then the corresponding $K3$ surface degenerates into a union of two rational surfaces intersecting along an elliptic curve (in "generic" cases). In this case the corresponding period is in the boundary of $(D/\Gamma)^*$. If B does not pass through the fixed points of I, then the corresponding degeneration of Enriques surface is a rational surface with a double curve along an elliptic curve. If B passes through a fixed point of I, it corresponds to a union of six rational surfaces, which consists of two rational surfaces S_1, S_2 intersecting transversally along a rational curve, and four $\boldsymbol{P}^2$'s, each of which intersects S_1 and S_2 like three coordinate planes in $\boldsymbol{C}^3$.

In these two cases, the period does not depend on the choice of the 1-parameter family which we use. This fact follows from the extension theorem of Borel [2] and others. The same extension theorem allows us to restrict our consideration to generic cases.

Finally suppose that B has a triple or quadruple component. Then that component is of the form $Y_1Z_2-\alpha Y_2Z_1=0$ with some constant α. These cases can be reduced to the cases of lower multiplicities by blowing up along the multiple component (cf. [3], §§10–12. But the situation is more transparent here than it was there). In the case of triple components, the corresponding periods lie in the closure of the union of the hypersurfaces H_e/Γ. The case of quadruple components corresponds to Enriques surfaces of special type and their degenerations.

References

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