12. On the Divisibility by 2 of the Eigenvalues of Hecke Operators

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(Communicated by Kunihiko KODAIRA, M. J. A., April 12, 1977)

Introduction. Prof. J.-P. Serre presented to us the following problem in his seminar which was held at the Research Institute for Mathematical Science of Kyoto University on the 27th of March in 1976 after the International Symposium on Algebraic Number Theory.

"Let $w+2$ be an even integer $\geqq 2$ and λ_p an eigenvalue of the Hecke operator $T(p)$ on cusp forms of weight $w+2$ for $SL(2, Z)$ where p is a rational prime. It is true that $\lambda_p/2$ is an algebraic integer? (It is true for $w+2 \leqq 96$ and all prime p.)"

In this paper we will give an affirmative answer for the above question. I continue my research in this direction now. I wish to express my gratitude heartily to Prof. J.-P. Serre for presenting this problem and informing me of many kind valuable suggestions on it and on the improvement of this paper.

§ 1. We denote by S_{w+2} the space of cusp forms of weight $w+2$ for the full modular group $SL(2, Z)$. We put $S^R_{w+2} = \{f \in S_{w+2} \mid$ all the Fourier coefficients of f at $z = i\infty$ are real numbers$\}$.

Theorem 1. *Any eigenvalue is divisible by* 2 *of the Hecke operator* $T(p)$ *on* S_{w+2} *for any rational prime* p *and any even weight* $w+2$.

Prof. J.-P. Serre informed me two statements equivalent to this theorem.

a) *In the space of cusp forms of any even weight* (*with real or rational coefficients*) *there is a lattice which is stable by the* $T(p)/2$. (*Such a lattice is not unique. But we can describe it nicely. See the proposition below.*)

b) *If* $a_{i,k}(p)$ *denotes the i-th coefficient of the characteristic polynomial of the* $T(p)$ *on* S_k, *we have* $2^i \mid a_{i,k}(p)$ *for any prime* p *and any even weight* k. (*We note that* $a_{i,k}(p)$ *is an integer which can easily be computed on machine.*)

Remark. We have extended this theorem to the case of real cusp forms of weight $w+2$ for some congruence subgroups (see [9], [10]).

Suggested by Prof. Serre, using the results obtained in this paper, I proved a more precise statement, the next Theorem 2, quite recently.

Theorem 2. *On cusp forms for* $SL(2, Z)$, *we have*

(i) *Any eigenvalue of the* $T(p)$ *is divisible by* 4 *for any prime* p *such that* $p \equiv -1 \bmod 4$ *and any weight* $w+2$.

(ii) (*Any eigenvalue of the* $T(p)$) -2 *is divisible by* 4 *for any prime* p *such that* $p\equiv +1 \bmod 4$ *and any weight* $w+2$.

Proof of Theorem 2 will appear somewhere.

Lemma 1 (Manin [4], Shimura [7]). *Shimura-Eichler Isomorphism. The map*

$$S^R_{w+2}\to \boldsymbol{R}^{w/2};\ f\mapsto\left(\int_0^{i\infty} f(z)z\,dz, \int_0^{i\infty} f(z)z^3dz, \cdots, \int_0^{i\infty} f(z)z^{w-1}dz\right)$$

is an injective $\boldsymbol{R}$*-linear mapping.*

Lemma 2 (Manin 7.6 [3] and Proposition 3.2 [4]). *Let* α *be a rational number such that* $0<\alpha<1/2$, *and let*

$$\alpha=\frac{b_n}{d_n}, \frac{b_{n-1}}{d_{n-1}}, \cdots, \frac{b_0}{d_0}=\frac{0}{1}$$

be the successive convergents to α *in irreducible form. For any* $f\in S_{w+2}$ *and any integer* l *with* $0\leqq l\leqq w$, *we have*

$$\int_0^{\alpha} f(z)z^l dz=\sum_{j=0}^{w} e^l_j \int_0^{i\infty} f(z)z^j dz$$

where each e^l_j *is equal to the rational integer*

$$-\sum_{k=1}^{n}\sum_{x=0}^{j}\binom{l}{x}\binom{w-l}{j-x} b^x_{k-1}b^{l-x}_k d^{j-x}_{k-1}d^{w-l-j+x}_k(-1)^{k(w-j)}.$$

§2. Proof of Theorem 1. Let $f(\neq 0)\in S_{w+2}$ be an eigenfunction of the Hecke operator $T(m)$ with an eigenvalue λ_m. We may assume that $f\in S^R_{w+2}$. For a polynomial function P of z, we denote by $v(P, dz)$ the $\boldsymbol{C}^{w/2}$ valued differential form given by

$$v(P, dz)={}^t(P(z)dz, P(z)^3dz, P(z)^5dz, \cdots, P(z)^{w-1}dz).$$

We also denote by M_p the $w/2\times w/2$ diagonal matrix whose (i, i) component is $p^{2i-1}+p^{w-2i+1}$.

Now let m be an odd prime p. By an elementary computation, we have

$$\begin{aligned}&\int_0^{i\infty} f\,|\,T(p)(z)v(z, dz)\\ (1)\qquad &=p^{-1}\sum_{b \bmod p}\int_0^{i\infty} f\left(\frac{z+p}{p}\right)v(z, dz)+p^{w+1}\int_0^{i\infty} f(pz)v(z, dz)\\ &=M_p\int_0^{i\infty} f(z)v(z, dz)+\sum_{b \bmod p\not\equiv 0}\int_{b/p}^{i\infty} f(z)v(pz-b, dz).\end{aligned}$$

We set

$$J(p, b, f)=\int_{b/p}^{i\infty} f(z)v(pz-b, dz).$$

Since $f\in S^R_{w+2}$, it is easy to see that $\overline{J(p, b, f)}=J(p, -b, f)$ where —— denotes the complex conjugation. From this we obtain that

$$\begin{aligned}&\sum_{b \bmod p\not\equiv 0} J(p, b, f)=\left(\sum_{b=1}^{(p-1)/2}+\sum_{b=-(p-1)/2}^{-1}\right)J(p, b, f)\\ (2)\qquad &=\sum_{b=1}^{(p-1)/2}(J(p, b, f)+\overline{J(p, b, f)})=\sum_{b=1}^{(p-1)/2} 2\,\mathrm{Re}\,J(p, b, f)\\ &=\sum_{b=1}^{(p-1)/2} 2\,\mathrm{Re}\left\{\int_0^{i\infty} f(z)v(pz-b, dz)-\int_0^{b/p} f(z)v(pz-b, dz)\right\}.\end{aligned}$$

Lemma 2 implies this { } is expressed as a $\boldsymbol{Z}$ linear combination of the periods

$$\int_0^{i\infty} f(z)z^l dz \qquad (l=0,1,\cdots,w)\ .$$

Since $f\in S_{w+2}^R$, odd periods (odd l) of f are real and even ones (even l) are pure imaginary. Taking real part of { }, only the odd periods remain in the right side of (2). Hence there exists a $w/2\times w/2$ matrix $A(p)$ with rational integer coefficients which satisfies

$$(2)=2A(p)\int_0^{i\infty} f(z)v(z,dz).$$

We put $B(p)=2^{-1}M_p+A(p)$, whose coefficients are all rational integers. Therefore (1) implies

$$0=(\lambda_p-2B(p))\int_0^{i\infty} f(z)v(z,dz).$$

Lemma 1 asserts that

$$\int_0^{i\infty} f(z)v(z,dz)\neq\vec{0}$$

if and only if $f(z)\neq 0$. Hence λ_p must be an eigenvalue of $2B(p)$.

Now let $m=2$. Then we have

$$\begin{aligned}&\int_0^{i\infty} f\,|\,T(2)(z)v(z,dz)\\ (1)'\qquad &=\int_0^{i\infty}\left(\frac{1}{2}f\left(\frac{z}{2}\right)+2^{w+1}f(2z)+\frac{1}{2}f\left(\frac{z+1}{2}\right)\right)v(z,dz)\\ &=M_2\int_0^{i\infty} f(z)v(z,dz)+\int_{1/2}^{i\infty} f(z)v(2z-1,dz).\end{aligned}$$

It is easy to see that there exists a $w/2\times w/2$ matrix $C(2)$ with rational integer coefficients which satisfies

$$\operatorname{Re}\int_0^{i\infty} f(z)v(2z-1,dz)=2C(2)\int_0^{i\infty} f(z)v(z,dz).$$

Set $\gamma=\begin{pmatrix}1&0\\2&1\end{pmatrix}$. Then $\gamma(0)=0$ and $\gamma(i\infty)=1/2$. We see also that

$$\begin{aligned}\operatorname{Re}\int_0^{1/2} f(z)v(2z-1,dz)&=\operatorname{Re}\int_{\gamma(0)}^{\gamma(i\infty)} f(z)v(2z-1,dz)\\ &=-\begin{pmatrix}0&&1\\&\cdot^{\cdot^{\cdot}}&\\1&&0\end{pmatrix}\operatorname{Re}\int_0^{i\infty} f(z)v(2z+1,dz)=2D(2)\int_0^{i\infty} f(z)v(z,dz),\end{aligned}$$

where $D(2)$ is a $w/2\times w/2$ matrix with rational integer coefficients. We put $B(2)=2^{-1}M_2+C(2)-D(2)$. Since the left side of (1)$'$ is in $\boldsymbol{R}^{w/2}$, we obtain

$$\text{the left side of }(1)'=2B(2)\int_0^{i\infty} f(z)v(z,dz).$$

We note that $B(2)$ is a $w/2\times w/2$ matrix with rational integer coefficients. Therefore if $m=2$, λ_2 is an eigenvalue of $2B(2)$.

§ 3. We put

$$L_{w+2}=\left\{f \in S_{w+2}^{R} \,\middle|\, \int_{0}^{i\infty} f(z)z^{l}dz \in \boldsymbol{Z} \text{ for all odd } l \text{ with } 1 \leqq l \leqq w-1\right\}.$$

Lemma 3 (Manin, Proposition 2.3 [4]). *The image of the map in Lemma 1 is given by Eichler-Shimura linear equations which have rational integer coefficients.*

By Lemmas 2 and 3 we can rephrase our proof of Theorem 1 as follows.

Proposition. *L_{w+2} is a lattice in S_{w+2}^{R} which is stable by the $T(p)/2$ where p is any rational prime.* (This Proposition is also informed by Prof. Serre.)

Further my proof of Theorem 2, shows that L_{w+2} *is stable by the* $T(p)/4$ *and the* $(T(q)-2)/4$ *for any prime* p, q *with* $p\equiv -1 \bmod 4$ *and* $q\equiv 1 \bmod 4$.

Remark. In the case of level 2 we have a similar result. We put

$$L_{w+2}(2)=\left\{f \mid f \text{ is a real cusp form of weight } w+2 \text{ on } \Gamma(2) \text{ such that } \int_{0}^{i\infty} (f\mid[g])(z)z^{l}dz \in \boldsymbol{Z} \text{ for all } g \in SL(2,\boldsymbol{Z}) \text{ and for all odd } l \text{ with } 1\leqq l \leqq w-1\right\}.$$

Then $L_{w+2}(2)$ is a lattice in real cusp forms on $\Gamma(2)$ which is stable by the $T'(p)/2$ for all prime p.

$$\begin{array}{ccc} S_{w+2}^{R} & \subset & S_{w+2}^{R}(\Gamma(2)) \\ \cup & & \cup \\ L_{w+2} & \subset & L_{w+2}(2). \end{array}$$

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