7. On the Relations between the Stability of Linear Systems and the Characteristic Roots of the Coefficient Matrix

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1. Introduction. Consider the homogeneous linear differential equation

(1) $\dot{x}=A(t)x \qquad (x: n\text{-vector})$

where the coefficient $n\times n$ matrix $A(t)$ is continuously differentiable in an interval $I=[0, +\infty)$. In this paper, we shall study the relations between the stability of the system (1) and the characteristic roots of the time-variant coefficient matrix $A(t)$, of which all characteristic roots are constant.

Throughout this paper, we use the vector Euclidean norm $\|x\| =\sqrt{\sum_{i=1}^{n}|x_i|^2}$ and the induced matrix norm $\|A\|=\sup_{\|x\|=1}\|Ax\|$.

Moreover, the definitions of stability, asymptotic stability and instability are the same as given in W. A. Coppel [1].

2. Theorem. We shall give a theorem, which was proved by the second author [3].

Theorem 1. *The homogeneous linear equation*

(1) $\dot{x}=A(t)x$

is reduced to the homogeneous linear equation

(2) $\dot{y}=B(t)y$

under the transformation

(3) $x=e^{St}y$

if and only if there exists a constant matrix S *satisfying the equations*

(4) $\dot{A}(t)=SA(t)-A(t)S-e^{St}\cdot\dot{B}(t)\cdot e^{-St}$

(5) $A(0)=S+B(0).$

3. Relations between the stability and the characteristic roots of $A(t)$. In the above Theorem 1, if we can choose a constant matrix B, we can express the fundamental matrix of the system (1) by the form

(6) $\Phi(t)=e^{St}\cdot e^{Bt}.$

In this case, the stability of the system (1) is completely decided by the characteristic roots of S and B, therefore is independent of ones of $A(t)$.

Let the coefficient matrix $A(t)$ be given in the following form:

$$A(t)=\begin{pmatrix} a+c\cos 2\omega t+d\sin 2\omega t & b+d\cos 2\omega t-c\sin 2\omega t \\ -b+d\cos 2\omega t-c\sin 2\omega t & a-c\cos 2\omega t-d\sin 2\omega t \end{pmatrix}. \tag{7}$$

Then we can apply Theorem 1, by choosing

$$S=\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{8}$$

$$B=\begin{pmatrix} a+c & b+d-\omega \\ -b+d+\omega & a-c \end{pmatrix}. \tag{9}$$

Now the characteristic equations of $A(t)$, S and B are

$$\det(\lambda I-A(t))=\lambda^2-2a\lambda+a^2+b^2-c^2-d^2, \tag{10}$$

$$\det(\lambda I-S)=\lambda^2+\omega^2, \tag{11}$$

$$\det(\lambda I-B)=\lambda^2-2a\lambda+a^2+b^2-c^2-d^2+\omega(\omega-2b) \tag{12}$$

respectively.

Set $2a=-11$, $b=d=6$, $2c=-9$ and $\omega=6$. Then the characteristic roots of $A(t)$ are -1, -10. On the other hand, the characteristic roots of S and B are $\pm 6i$ and 2, -13 respectively. Therefore the system (1) is unstable. (This example is due to R. E. Vinogradov [4].)

If we choose suitable values for a, b, c, d and ω, we can get some examples which show the relations between the stability of the system (1) and the characteristic roots of $A(t)$. The results are given in Table I below. We shall denote the real parts of the characteristic roots of $A(t)$ by λ_1, λ_2 ($\lambda_1 \leqq \lambda_2$).

Table I

Char. roots of $A(t)$	Unstable	Stable not asymp. sta.	Asymptotically stable
$\lambda_1 \leqq \lambda_2 < 0$	$2a=-11$, $2c=-9$ $b=d=6$, $\omega=6$	$a=-2$, $d=0$ $b=c=\omega=2$	$a=-1$ $b=c=d=\omega=0$
$0<\lambda_1 \leqq \lambda_2$	$a=1$ $b=c=d=\omega=0$	*	*
$\lambda_1<0<\lambda_2$	$a=b=1$, $c=2$ $d=\omega=0$	$a=-2$, $b=0$ $c=d=\omega=2$	$2a=-11$, $b=c=0$ $2d=15$, $\omega=6$
$\lambda_1=\lambda_2=0$	$a=d=0$ $b=c=\omega=2$	$a=d=0$ $b=-1$, $c=\omega=1$	*
$\lambda_1<0=\lambda_2$	$a=-1$ $b=c=d=\omega=1$	$a=-1$, $\omega=2$ $b=c=d=1$	$a=b=c=d=-1$ $\omega=1$
$\lambda_1=0<\lambda_2$	$a=c=1$ $b=d=\omega=0$	*	*

Remark. In the above Table I, for example, the notation * in the second row under the third column implies that we can not find a, b, c, d

and ω such that $0<\lambda_1\leqq\lambda_2$ and the system (1) is asymptotically stable. This is also a direct result from the Jacobi-Liouville's formula

$$|\det \Phi(t)|=\left|\exp\int_0^t trA(s)\,ds\right| = \exp\int_0^t(\lambda_1+\lambda_2)ds. \tag{13}$$

This formula implies that the notation * in Table I is always true for any 2×2 matrix $A(t)$, of which all characteristic roots are constant.

4. Relations between the stability and the characteristic roots of $H(t)=(A(t)+A^*(t))/2$. Table I shows that there are few relations between the stability of the system (1) and the characteristic roots of $A(t)$. On the other hand, we know the other criteria based on the (real) characteristic roots of the Hermitian matrix $H(t)=(A(t)+A^*(t))/2$.

Lemma [1]. *Let $x(t)$ be a solution of the system* (1). *Then $x(t)$ satisfies the following inequalities*:

$$\|x(0)\|\cdot\exp\left(\int_0^t\lambda(\tau)d\tau\right)\leqq\|x(t)\|\leqq\|x(0)\|\cdot\exp\left(\int_0^t\Lambda(\tau)d\tau\right), \tag{14}$$

where $\lambda(t)$, $\Lambda(t)$ are the smallest characteristic root of $H(t)$ and the largest characteristic root of $H(t)$ respectively.

In this section, we shall examine details of inequalities of (14) in the case $n=2$.

Let a coefficient matrix $A(t)$ be as the same as (7). Then the characteristic equation of $H(t)=(A(t)+A^*(t))/2$ is

Table II

Char. roots of $H(t)$	Unstable	Stable not asymp. sta.	Asymptotically stable
$\mu_1\leqq\mu_2<0$	*	*	◎
$0<\mu_1\leqq\mu_2$	◎	*	*
$\mu_1<0<\mu_2$	$a=b=d=\omega=0$ $c=-1$	$a=-2,\ b=0$ $c=d=\omega=2$	$2a=-11,\ 2d=15$ $b=c=0,\ \omega=6$
$\mu_1=\mu_2=0$	*	◎	*
$\mu_1<0=\mu_2$	*	$a=-5,\ b=\omega=2$ $c=3,\ d=4$	$a=-5,\ b=5$ $c=3,\ d=4,\ \omega=2$
$\mu_1=0<\mu_2$	$2a=1,\ 2c=-1$ $b=d=\omega=0$	**	*

(15) $$\det(\lambda I - H(t)) = \lambda^2 - 2a\lambda + a^2 - c^2 - d^2.$$

We shall denote the (real) characteristic roots of $H(t)$ by μ_1, μ_2 $(\mu_1 \leqq \mu_2)$. Selecting the suitable values for a, b, c, d and ω, we get relations between the stability of the system (1) and the characteristic roots of $H(t)$. The results are given in Table II.

Remark. In the above Table II, the notation * is the same as remarked one under Table I. The notation ◎ implies that if μ_1, μ_2 satisfy the row condition, then the system (1) has always the column headed property. These are proved directly by Lemma.

The notation ** implies that there is no matrix $A(t)$ such that $\mu_1 = 0 < \mu_2$ and the corresponding system (1) is stable but not asymptotically stable. If there is such a system, then the induced matrix norm of the fundamental matrix $\Phi(t)$ must be bounded. On the other hand, from the formula (13)

$$\begin{aligned} |\det \Phi(t)| &= \exp \int_0^t trA(s)ds \\ &= \exp \int_0^t trH(s)ds \\ &= \exp(\mu_2 t). \end{aligned}$$

This contradicts the boundedness of $\Phi(t)$.

Thus, in Table II, the notations *, **, ◎ are always true for any 2×2 matrix $A(t)$, of which all characteristic roots are constant.

References

[1] W. A. Coppel: Stability and Asymptotic Behavior of Differential Equations. D. C. Heath, Boston (1965).

[2] M. Y. Wu: An extension of "A new method of computing the state transition matrix of linear time-varying systems". IEEE Trans. Automat. Contr., AC-**19**, 619–620 (1974).

[3] M. Yamamoto: On the homogeneous linear systems of differential equations with variable coefficients. Proc. Japan Acad., **53** (A), 17–20 (1977).

[4] V. I. Zubov: Mathematical Methods for the Study of Automatic Control Systems. Pergamon Press (1962).