55. On Families of Effective Divisors on Algebraic Manifolds

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1. By an algebraic manifold, we mean a connected compact complex manifold imbedded in a complex projective space. Let V be an algebraic manifold. We denote by $\mathrm{Pic}^0(V)$ the set of all holomorphic line bundles on V whose Chern classes vanish. It is well known that $\mathrm{Pic}^0(V)$ is an abelian variety of dimension $q=\dim H^1(V, \mathcal{O})$, the irregularity of V. Let $c \in H^2(V, \boldsymbol{Z})$ be a cohomology class of type (1, 1). We put

$$\boldsymbol{D}^c(V)=\{D \mid D \text{ is an effective divisor on } V \text{ with } c([D])=c\},$$

where $[D]$ is the line bundle determined by D and $c([D])$ is the Chern class of $[D]$. According to Weil [7] (see also Kodaira [3]), $\boldsymbol{D}^c(V)$ is a projective variety (i.e., a complex space imbedded in a complex projective space) and the Jacobi mapping

$$\Phi : D \in \boldsymbol{D}^c(V) \rightarrow [D-D_o] \in \mathrm{Pic}^0(V)$$

is holomorphic, where $D_o \in \boldsymbol{D}^c(V)$ is a fixed effective divisor.

In this note, we state the following theorems. Details will be published elsewhere.

Theorem 1. *Assume that there is an effective divisor $D \in \boldsymbol{D}^c(V)$ such that*

$$\dim H^0(V, \mathcal{O}([D])) > \dim H^1(V, \mathcal{O}([D])).$$

Then the Jacobi mapping Φ is surjective and each fiber of Φ has dimension at least $\dim H^0(V, \mathcal{O}([D])) - \dim H^1(V, \mathcal{O}([D])) - 1$.

In general, we put $W_c=\Phi(\boldsymbol{D}^c(V))$. It is a closed subvariety of $\mathrm{Pic}^0(V)$.

Theorem 2. *For an effective divisor $D \in \boldsymbol{D}^c(V)$, put $a=\dim H^0(V, \mathcal{O}([D]))$ and $b=\dim H^1(V, \mathcal{O}([D]))$. Assume that $a \leqq b$. Then there are an open neighbourhood U of $x=\Phi(D)$ in $\mathrm{Pic}^0(V)$ and a $(a \times b)$-matrix valued holomorphic function $A(y)$, $y \in U$, on U such that $W_c \cap U$ is the set of zeros of all $a \times a$ minors of $A(y)$.*

Remark. If V is a non-singular curve, then Theorem 1 reduces to Jacobi inversion problem and Theorem 2 reduces to Kempf's theorem [5].

2. Theorem 1 and Theorem 2 are easy consequences of the fol-

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lowing theorems. In the sequel, by a complex space, we mean a reduced, Hausdorff, complex analytic space.

Theorem 3. *Let $\{V_s\}_{s\in S}$ be a family of compact complex manifolds with the parameter space S, a complex space. Let $\{F_s\}_{s\in S}$ be a family of holomorphic vector bundles over $\{V_s\}_{s\in S}$. Then, for each point $o\in S$, there are an open neighbourhood S' of o in S and a vector bundle homomorphism*

$$u: H^0(V_o, \mathcal{O}(F_o))\times S' \to H^1(V_o, \mathcal{O}(F_o))\times S'$$

such that the union $\bigcup_{s\in S'} H^0(V_s, \mathcal{O}(F_s))$ is identified with the kernel of u.

The following Theorem 4 and Theorem 5 are considered as special cases of Schuster [6].

Theorem 4. *Let $\{V_s\}_{s\in S}$ and $\{F_s\}_{s\in S}$ be as in Theorem 3. Then the union $\boldsymbol{H}=\bigcup_{s\in S} H^0(V_s, \mathcal{O}(F_s))$ admits a complex space structure so that $(\boldsymbol{H}, \lambda, S)$ is a complex linear space in the sense of Grauert* [1], *where $\lambda: \boldsymbol{H}\to S$ is the canonical projection.*

Theorem 5. *Let $\{V_s\}_{s\in S}$ and $\{F_s\}_{s\in S}$ be as in Theorem 3. Let $P(F_s)$ be the projective space associated with $H^0(V_s, \mathcal{O}(F_s))$. ($P(F_s)$ is empty if $H^0(V_s, \mathcal{O}(F_s))=0$.) Then the union $\boldsymbol{P}=\bigcup_{s\in S} P(F_s)$ admits a complex space structure so that the canonical projection $\mu: \boldsymbol{P}\to S$ is a proper holomorphic mapping.*

By Theorem 3, we easily get

Theorem 6. *Let $\{V_s\}_{s\in S}$, $\{F_s\}_{s\in S}$ and $\mu: \boldsymbol{P}\to S$ be as in Theorem 5. For a point $o\in S$, assume that $\dim H^0(V_o, \mathcal{O}(F_o)) > \dim H^1(V_o, \mathcal{O}(F_o))$. Then there is an open neighbourhood S' of o in S such that $\mu' = \mu|\mu^{-1}(S'): \mu^{-1}(S')\to S'$ is surjective and each fiber of μ' has dimension at least $\dim H^0(V_o, \mathcal{O}(F_o)) - \dim H^1(V_o, \mathcal{O}(F_o)) - 1$.*

Theorem 7. *Let $\{V_s\}_{s\in S}$, $\{F_s\}_{s\in S}$ and $\mu: \boldsymbol{P}\to S$ be as in Theorem 5. For a point $o\in S$, put $a=\dim H^0(V_o, \mathcal{O}(F_o))$ and $b=\dim H^1(V_o, \mathcal{O}(F_o))$. Assume that $a\leqq b$. Then there are an open neighbourhood S' of o in S and a $(a\times b)$-matrix valued holomorphic function $A(s)$, $s\in S'$, on S' such that $\mu(\boldsymbol{P})\cap S'$ is the set of zeros of all $a\times a$ minors of $A(s)$.*

3. In order to get Theorem 1 and Theorem 2 from the theorems in § 2, we consider the case

$$S=\mathrm{Pic}^0(V),$$
$$V_x = V \text{ (fixed)},$$
$$F_x = B_x\otimes[D_o],$$

where B_x is the line bundle, with the Chern class 0, corresponding to the point $x\in \mathrm{Pic}^0(V)$. Then we can easily prove that the complex space $\boldsymbol{P}$ in Theorem 5 is canonically biholomorphic to $\boldsymbol{D}^c(V)$. Now, Theorem 6 and Theorem 7 reduce to Theorem 1 and Theorem 2, respectively.

4. Let $x \in \mathrm{Pic}^0(V)$. Let

$$\sigma : H^0(V, \mathcal{O}(F_x)) \times H^1(V, \mathcal{O}) \to H^1(V, \mathcal{O}(F_x))$$

be the bilinear map defined by

$$\sigma(\xi, h)_{ik}(z) = \xi_i(z) h_{ik}(z),$$

where $\xi = \{\xi_i(z)\} \in H^0(V, \mathcal{O}(F_x))$ and $h = \{h_{ik}(z)\} \in H^1(V, \mathcal{O})$ for a suitable Stein open covering $\{U_i\}$ of V. We put

$$\sigma(\xi, h) = \sigma_\xi(h) = \sigma(h)(\xi)$$

by abuse of notation. Let $D = (\xi)$ be the zero divisor of ξ. D is said to be *semi-regular* if and only if the linear map $\sigma_\xi : H^1(V, \mathcal{O}) \to H^1(V, \mathcal{O}(F_x))$ is surjective. Note that if V is a non-singular curve, then every D is semi-regular.

The semi-regularity theorem by Kodaira-Spencer [4] says that if D is semi-regular, then D is a non-singular point of $\boldsymbol{D}^c(V)$ and

$$\dim_D \boldsymbol{D}^c(V) = \dim H^0(V, \mathcal{O}([D])) - \dim H^1(V, \mathcal{O}([D])) + q - 1.$$

We note that the differential at (ξ, x) of the mapping u in Theorem 3 is equal to

$$\begin{pmatrix} 0 & \sigma_\xi \\ 0 & 1 \end{pmatrix}$$

in our case. From this fact, we get the semi-regularity theorem.

Finally, we generalize Kempf's theorems [2] as follows: For a point $x \in \mathrm{Pic}^0(V)$, assume that every divisor in $\Phi^{-1}(x)$ is semi-regular. Let N and C be the normal bundle of $\boldsymbol{D}^c(V)$ along $\Phi^{-1}(x)$ and the tangent cone of $W_c = \Phi(\boldsymbol{D}^c(V))$ at x, respectively. Let

$$\rho : N \to C$$

be the mapping induce by Φ.

Kempf's theorem for algebraic manifolds (c.f. Kempf [2]). *For a point $x \in \mathrm{Pic}^0(V)$, assume that every divisor in $\Phi^{-1}(x)$ is semi-regular. Assume moreover that*

$$\dim H^0(V, \mathcal{O}(F_x)) \leqq \dim H^1(V, \mathcal{O}(F_x)) + 1.$$

Then

(1) *$\rho : N \to C$ is a rational resolution.*

(2) *The degree of C is the binomial coefficient*

$$\binom{\dim H^1(V, \mathcal{O}(F_x))}{\dim H^0(V, \mathcal{O}(F_x)) - 1}.$$

(3) *If $\dim H^0(V, \mathcal{O}(F_x)) \leqq \dim H^1(V, \mathcal{O}(F_x))$, then the ideal defining C is generated by the maximal minors of the matrix valued function $\sigma(h)$ on $H^1(V, \mathcal{O})$.*

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