52. The Paley-Wiener Type Theorem for Finite Covering Groups of $SU(1,1)$

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An n-fold covering group G of $SU(1,1)$ is realized as $G=\{(\gamma,\omega);\ \gamma\in \boldsymbol{C}, |\gamma|<1, \omega\in \boldsymbol{R}/2n\pi\boldsymbol{Z}\}$ with the multiplication: $(\gamma,\omega)(\gamma',\omega')=(\gamma'',\omega'')$, where $\gamma''=(\gamma e^{-2i\omega'}+\gamma')(1+\gamma\bar{\gamma}'e^{-2i\omega'})^{-1}$, and

$$\omega''\equiv\omega+\omega'+(2i)^{-1}\log(1+\gamma\bar{\gamma}'e^{-2i\omega'})(1+\bar{\gamma}\gamma'e^{2i\omega'})^{-1} \qquad (\mathrm{mod}\ 2n\pi),$$

and we take the principal branch of logarithm. Put $u_\theta=(0,-\theta/2)$, $a_t=(\mathrm{th}\,(t/2),0)$. Then each element $g\in G$ can be expressed as $g=u_\varphi a_t u_\psi$ $(0\le\varphi<4n\pi,\ t\ge 0,\ 0\le\psi<2\pi)$.

§ 1. Let $d\mu(\zeta)$ be the ordinary normalized Haar measure on the unit circle $\boldsymbol{T}$ in $\boldsymbol{C}$ and put $\mathfrak{H}=L^2(\boldsymbol{T};d\mu(\zeta))$. For any integer k with $-n+1\le k\le n$ and $s\in\boldsymbol{C}$, we define operators $U^k(g,s)(g\in G)$ by

$$U^k(g,s)f(\zeta)=e^{-2i\omega\lambda_k}\left[\frac{1+\bar{\gamma}\zeta}{1+\gamma\bar{\zeta}}\right]^{\lambda_k}(1-|\gamma|^2)^{1/2+s}\,|1+\bar{\gamma}\zeta|^{-1-2s}f\left(e^{2i\omega}\zeta\frac{1+\gamma\bar{\zeta}}{1+\bar{\gamma}\zeta}\right),$$

where $\lambda_k=k/2n$, $g^{-1}=(\gamma,\omega)$, $\zeta\in\boldsymbol{T}$ and $f\in\mathfrak{H}$. Then $g\mapsto U^k(g,s)$ is a strongly continuous bounded representation of G for any fixed $s\in\boldsymbol{C}$. We put $e_p(\zeta)=\zeta^{-p}$ $(p\in\boldsymbol{Z})$. Clearly $\{e_p;\ p\in\boldsymbol{Z}\}$ forms a C.O.N.S. in $\mathfrak{H}$.

Let $\alpha_p^k(s)(-n+1\le k\le n, p\in\boldsymbol{Z})$ be a rational function defined by $\alpha_p^k(s)$

$$=\Gamma\left(\frac{1}{2}+\lambda_k+s\right)\Gamma\left(\frac{1}{2}+\lambda_k-s\right)^{-1}\Gamma\left(p+\frac{1}{2}+\lambda_k-s\right)\Gamma\left(p+\frac{1}{2}+\lambda_k+s\right)^{-1}.$$

We can define for $\mathrm{Re}\,s\ge 0$ a bounded operator $A^k(s)$ on $\mathfrak{H}$ by $A^k(s)e_p=\alpha_p^k(s)e_p$.

Lemma 1.

$$A^k(s)U^k(g,s)=U^k(g,-s)A^k(s) \qquad (g\in G,\ \mathrm{Re}\,s\ge 0).$$

Let $\mathfrak{H}_j^+=\sum_{p\ge j}^{\oplus}\boldsymbol{C}e_p$ and $\mathfrak{H}_j^-=\sum_{p\le -j}^{\oplus}\boldsymbol{C}e_p$. Then we have

Lemma 2. *$\mathfrak{H}_j^\varepsilon$ is $U^k\left(\cdot,\varepsilon\lambda_k+j-\frac{1}{2}\right)$-invariant* ($\varepsilon=+,-$ *and* $j=1, 2,\cdots$).

Using Lemma 2, we can construct other representations $V^\pm(\cdot,j)$ of G, which are unitary under certain inner product and irreducible (discrete series, except for $(\varepsilon,j)=(-,1)$).

§ 2. Put $u_{pq}^k(g,s)=(U^k(g,s)e_q,e_p)$. Using Lemma 1, we have for any $s\in\boldsymbol{C}$, $u_{pq}^k(g,-s)=\Lambda_{pq}^k(s)u_{pq}^k(g,s)$, where $\Lambda_{pq}^k(s)=\alpha_p^k(s)/\alpha_q^k(s)$. The matrix elements $v_{pq}^{k,\pm}(g,j)$ of $V^\pm(\cdot,j)$ are given as follows: for "$p,q\ge j$

when $\varepsilon=+$" or "$p, q\leq -j$ when $\varepsilon=-$", $v_{pq}^{k,\varepsilon}(g, j)=\omega_{pq}^{k,\varepsilon}(j)u_{pq}^{k}\left(g, \varepsilon\lambda_k+j-\frac{1}{2}\right)$, where

$$\omega_{pq}^{k,\varepsilon}(j)=\prod_{0\leq l\leq \varepsilon q-j-1}\left[\frac{l+2(j+\varepsilon\lambda_k)}{l+1}\right]^{1/2}\cdot\prod_{0\leq l\leq \varepsilon p-j-1}\left[\frac{l+1}{l+2(j+\varepsilon\lambda_k)}\right]^{1/2}.$$

For the sake of convenience, we put $\omega_{pq}^{k,\varepsilon}(j)=0$ for any other triplet in the above definition.

§3. Let $\mathcal{D}_T$ be a Fréchet space of functions f on G such that $f(u_\varphi a_t u_\psi)=0$ for $t\geq T$, which is topologized as usual. Let $\mathcal{D}_T^k$ be a closed subspace of $\mathcal{D}_T$ consisting of functions f such that $f(u_{2\pi}g)=e^{ik\pi/n}f(g)$. Notice that $u_{2\pi}$ is a generator of the center of G.

Lemma 3. $\mathcal{D}_T=\sum_{-n+1\leq k\leq n}^{\oplus}\mathcal{D}_T^k$.

The "Fourier transform" of $f\in\mathcal{D}_T^k$ is the operator-valued function $\mathcal{F}(s)=\int f(g)U^k(g, s)dg$ $(s\in \boldsymbol{C})$. Let $\boldsymbol{N}$ be the set of all positive integers and put, according as $k\neq n$ or $k=n$ respectively,

$$N_{pq}^k=\left\{\lambda_k+j-\frac{1}{2}\,;\, j\in \boldsymbol{N} \text{ with } p<j\leq q\right\}$$
$$\cup\left\{-\lambda_k+j-\frac{1}{2}\,;\, j\in \boldsymbol{N} \text{ with } q\leq -j<p\right\},$$
$$N_{pq}^n=\{j\,;\, j\in \boldsymbol{N}\cup\{0\} \text{ with } p<j\leq q\}$$
$$\cup\{j\,;\, j\in \boldsymbol{N}\cup\{0\} \text{ with } q\leq -j-1<p\}.$$

Let $\mathcal{H}_T^k$ be the totality of bounded operator-valued entire functions $\mathcal{F}(s)$ on $\boldsymbol{C}$ which satisfy the following:

(i) for every non-negative integer r, there exists a constant C_r such that $\|\mathcal{F}(s)\|\leq C_r(1+|s|)^{-r}e^{T|\mathrm{Re}\, s|}$;

(ii) $(\mathcal{F}(-s)e_q, e_p)=\Lambda_{pq}^k(s)(\mathcal{F}(s)e_q, e_p)$ $(p, q\in \boldsymbol{Z})$;

(iii) $(\mathcal{F}(s)e_q, e_p)=0$ for all $s\in N_{pq}^k$;

(iv) for every quintet β of non-negative integers $\beta=(a, b, c, r, M)$, define $|\mathcal{F}|_\beta$ as below. Then $|\mathcal{F}|_\beta<\infty$:

$$|\mathcal{F}|_\beta=\sup_{p,q\in \boldsymbol{Z};\, j\in \boldsymbol{N}}\ \sup_{|\mathrm{Re}\, s|\leq M}(1+|p|)^a(1+|q|)^b\Big[(1+|s|)^r|(\mathcal{F}(s)e_q, e_p)|$$
$$+j^c\sum_{\varepsilon=+,-}\omega_{pq}^{k,\varepsilon}(j)\left|\left(\mathcal{F}\left(\varepsilon\lambda_k+j-\frac{1}{2}\right)e_q, e_p\right)\right|\Big].$$

Theorem. *Let us topologize $\mathcal{H}_T^k$ by means of the family of semi-norms $|\mathcal{F}|_\beta$. Then the Fourier transform $\mathfrak{F}: f\mapsto\mathcal{F}(\cdot)=\int f(g)U^k(g, \cdot)dg$ gives a topological isomorphism of $\mathcal{D}_T^k$ onto $\mathcal{H}_T^k$.*

§4. **Outline of the proof of Theorem.** We decompose $f\in\mathcal{D}_T^k$ into functions of different "K-type" for $K=\{u_\theta\,;\,\theta\in \boldsymbol{R}\}$. Let $\mathcal{D}_{pq,T}^k$ be the closed subspace of functions h such that

(1) $$h(u_\varphi g u_\psi)=\exp(i(p+\lambda_k)\varphi)h(g)\exp(i(q+\lambda_k)\psi).$$

Lemma 4. *Let f be a C^∞-function on G such that $f(u_{2\pi}g)=e^{ik\pi/n}f(g)$. Then f can be decomposed as*

$$f(g)=\sum_{p,q\in Z} f_{pq}(g) \qquad (\textit{pointwise absolute convergence}),$$

where f_{pq} satisfies (1).

In view of Lemma 4, we first investigate the case $f\in\mathcal{D}^k_{pq,T}$ separately. This turns out to study $\int f(g)u^k_{pq}(g,s)dg$. The case $\mathcal{D}^k_{00,T}$ is the most important, and the other cases can be reduced to this case in a similar way as in Part II of [1]. For the case of $\mathcal{D}^k_{00,T}$, we improve the method in Part I of [1], by giving an exact estimate of the growth of matrix elements at infinity. Once the Paley-Wiener type theorem for $\mathcal{D}^k_{pq,T}$ is established, our theorem follows by summing it up over p, q.

The details will be published elsewhere.

References

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