50. Nonlinear Parabolic Variational Inequalities with Time-dependent Constraints

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Let H be a (real) Hilbert space with the inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$ in H, and X a uniformly convex Banach space with the strictly convex dual X^*, natural pairing $(\cdot,\cdot)_X : X^*\times X\to R^1$ and with norm $\|\cdot\|_X$ in X. Suppose that X is a dense subspace of H and the natural injection from X into H is continuous. Then, identifying H with its dual in terms of the inner product $(\cdot,\cdot)_H$, we have the relation $X\subset H\subset X^*$ where H is dense in X^*. Let $0<T<\infty$ and $2\leqq p<\infty$ with $1/p+1/p'=1$, and put $\mathcal{H}=L^2(0,T;H)$ and $\mathcal{X}=L^p(0,T;X)$ with $\mathcal{X}^*=L^{p'}(0,T;X^*)$; the natural pairing between $\mathcal{X}^*$ and $\mathcal{X}$ is denoted by $(\cdot,\cdot)_{\mathcal{X}}$ as well.

We are given a family $\{K(t);0\leqq t\leqq T\}$ of closed convex subsets of X satisfying that

(KI) *for each* $r\geqq 0$ *there are real-valued functions* $\alpha_r\in W^{1,2}(0,T)$ *and* $\beta_r\in W^{1,1}(0,T)$ *with the following property: for each* $s,t\in[0,T]$ *with* $s\leqq t$ *and* $z\in K(s)$ *with* $\|z\|_H\leqq r$ *there exists* $z_1\in K(t)$ *such that*

$$\|z_1-z\|_H\leqq|\alpha_r(t)-\alpha_r(s)|(1+\|z\|_X^{p/2})$$

and

$$\|z_1\|_X^p-\|z\|_X^p\leqq|\beta_r(t)-\beta_r(s)|(1+\|z\|_X^p).$$

We put K_H=the closure of $K(0)$ in H and $\mathcal{K}=\{v\in\mathcal{X};v(t)\in K(t)$ for a.e. $t\in[0,T]\}$.

We are also given a family $\{A(t);0\leqq t\leqq T\}$ of (nonlinear) operators from $D(A(t))=X$ into X^* such that

(AI) $\mathcal{A}$ *defined by* $[\mathcal{A}v](t)=A(t)v(t)$ *is an operator from* $D(\mathcal{A})=\mathcal{X}$ *into* $\mathcal{X}^*$ *and maps bounded subsets of* $\mathcal{X}$ *into bounded subsets of* $\mathcal{X}^*$;

(AII) *for each* $h\in\mathcal{X}$ *there are a positive number* c_0 *and a function* $c_1\in L^1(0,T)$ *satisfying*

$$(A(t)z,z-h(t))_X\geqq c_0[z]_X^p-c_1(t)\qquad a.e.\ on\ [0,T]$$

for all $z\in X$, *where* $[\cdot]_X$ *is a seminorm on* X *such that* $[\cdot]_X+\|\cdot\|_H$ *gives a norm on* X *equivalent to* $\|\cdot\|_X$.

With the above notation, given $f\in\mathcal{X}^*$ and $u_0\in K_H$, our problem $(V_s;f,u_0)$ is to find a function $u\in\mathcal{K}$ such that

(i) $u'(=du/dt)\in\mathcal{X}^*$ and $(u'+\mathcal{A}u-f,u-v)_{\mathcal{X}}\leqq 0$ for all $v\in\mathcal{K}$;

(ii) $u(0)=u_0$ (note that $u\in C([0,T];H)$ if $u\in\mathcal{K}$ and $u'\in\mathcal{X}^*$).

This is the strong formulation, while in its weak formulation $(V_w; f, u_0)$, instead of (i) and (ii), only the following (iii) is required:

(iii) $(v'+\mathcal{A}u-f, u-v)_{\mathcal{X}}-\|u_0-v(0)\|_H^2/2\leqq 0$ for all $v\in\mathcal{K}$ with $v'\in\mathcal{X}^*$.

Our object is to show the existence and uniqueness of a solution to $(V_w; f, u_0)$. For this purpose we introduce the following (possibly multivalued) operator $\mathcal{L}_{u_0}$: $[u, g]\in G(\mathcal{L}_{u_0})$ (the graph of $\mathcal{L}_{u_0}$) if and only if $u\in\mathcal{K}$, $g\in\mathcal{X}^*$ and $(v'-g, u-v)_{\mathcal{X}}-\|u_0-v(0)\|_H^2/2\leqq 0$ for all $v\in\mathcal{K}$ with $v'\in\mathcal{X}^*$. As is easily checked, u is a solution to $(V_w; f, u_0)$ if and only if it is a solution of the functional equation $\mathcal{L}_{u_0}u+\mathcal{A}u\ni f$.

Now given $u_0\in K_H$, we consider the following operator $\mathcal{L}_{u_0}^s$ corresponding to the strong formulation of our problem: $[u, g]\in G(\mathcal{L}_{u_0}^s)$ if and only if $u\in\mathcal{K}$ with $u'\in\mathcal{X}^*$, $u(0)=u_0$, $g\in\mathcal{X}^*$ and $(u'-g, u-v)_{\mathcal{X}}\leqq 0$ for all $v\in\mathcal{K}$. Clearly, $\mathcal{L}_{u_0}$ is an extension of $\mathcal{L}_{u_0}^s$. Also, applying the results in [8] and [10], we can prove:

Theorem 1. (i) *For each $u_0\in K_H$, $\mathcal{L}_{u_0}$ is maximal monotone.*

(ii) *If $u_0\in K_H$ and $u\in D(\mathcal{L}_{u_0})$, then $u\in C([0, T]; H)$ and $u(0)=u_0$.*

(iii) *Let $u_{0,i}\in K_H$ and $[u_i, g_i]\in G(\mathcal{L}_{u_{0,i}})$ $(i=1, 2)$. Then for any s, $t\in[0, T]$ with $s\leqq t$,*

$$\|u_1(t)-u_2(t)\|_H^2-\|u_1(s)-u_2(s)\|_H^2\leqq 2\int_s^t (g_1-g_2, u_1-u_2)_X dr.$$

(iv) *Let $\{u_{0,n}\}\subset K_H$ and $\{[u_n, g_n]\}$ with $[u_n, g_n]\in G(\mathcal{L}_{u_{0,n}})$ be sequences such that $u_{0,n}\to u_0$ strongly in H, $u_n\to u$ strongly (resp. weakly) in $\mathcal{X}$ and $g_n\to g$ weakly (resp. strongly) in $\mathcal{X}^*$ as $n\to\infty$. Then $[u, g]\in G(\mathcal{L}_{u_0})$ and $u_n\to u$ strongly in $C([0, T]; H)$ as $n\to\infty$.*

(v) *Let $[u, g]\in G(\mathcal{L}_{u_0})$ with $u_0\in K_H$ and $\{u_{0,n}\}$ be a sequence in $K(0)$ such that $u_{0,n}\to u_0$ strongly in H as $n\to\infty$. Then there is a sequence $\{[u_n, g_n]\}$ such that $[u_n, g_n]\in G(\mathcal{L}_{u_{0,n}}^s)$, $g_n\in\mathcal{H}$, $g_n\to g$ weakly in $\mathcal{X}^*$ and $u_n\to u$ strongly both in $C([0, T]; H)$ and in $\mathcal{X}$ as $n\to\infty$.*

In addition to the assumptions we have made so far, assume that

(KII) *$z_1+z_2-z_3\in K(t)$ for any $z_1, z_2, z_3\in K(t)$ and $t\in[0, T]$.*

Then the following holds.

Proposition. *For each $u_0\in K_H$, $G(\mathcal{L}_{u_0})$ is convex and closed in $\mathcal{X}\times\mathcal{X}^*$ (and hence it is closed in the weak-weak topology of $\mathcal{X}\times\mathcal{X}^*$).*

Next, by using the above results and a slightly modified version of [6; Theorem 2], concerning the equation $\mathcal{L}_{u_0}u+\mathcal{A}u\ni f$, we have:

Theorem 2. *Suppose that $\mathcal{A}$ is of type M* (cf. [2]). *Then for each $u_0\in K_H$, the range of $\mathcal{L}_{u_0}+\mathcal{A}$ is the whole of $\mathcal{X}^*$, that is, the equation $\mathcal{L}_{u_0}u+\mathcal{A}u\ni f$ has a solution for every $f\in\mathcal{X}^*$.*

Theorem 3. *If there is a function $\omega\in L^1(0, T)$ such that $(A(t)z_1-A(t)z_2, z_1-z_2)_X+\omega(t)\|z_1-z_2\|_H^2\geqq 0$ for all $z_1, z_2\in K(t)$ and a.e. $t\in[0, T]$, then the equation $\mathcal{L}_{u_0}u+\mathcal{A}u\ni f$ admits at most one solution for each*

$u_0 \in K_H$ and $f \in \mathcal{X}^*$, and the solution depends continuously on u_0 and f.

The detailed proofs of the results mentioned above and their applications will be given in [9].

Remarks. (i) In Theorem 2, if $\mathcal{A}$ is a pseudo-monotone operator (cf. [2]) from $\mathcal{X}$ into $\mathcal{X}^*$, then the same conclusion of Theorem 2 is valid without the assumption (KII) (cf. [3; Théorème 1]).

(ii) Given $f \in \mathcal{X}^*$, consider the variational problem, with periodic condition, to find $u \in \mathcal{K}$ such that $(v' + \mathcal{A}u - f, u - v)_{\mathcal{X}} \leqq 0$ for all $v \in \mathcal{K}$ with $v' \in \mathcal{X}^*$ and $v(0) = v(T)$. To this problem the same type of treatment is available; in this case, we require that $K(T) \subset K(0)$, and the operator $\mathcal{L}_p$ corresponding to $\mathcal{L}_{u_0}$ is defined by the following: $[u, g] \in G(\mathcal{L}_p)$ if and only if $u \in \mathcal{K}$, $g \in \mathcal{X}^*$ and $(v' - g, u - v)_{\mathcal{X}} \leqq 0$ for all $v \in \mathcal{K}$ with $v' \in \mathcal{X}^*$ and $v(0) = v(T)$. For details, see [9].

(iii) In case $K(t)$ is time-independent, we find many interesting results on the problem $(V_s; f, u_0)$ or $(V_w; f, u_0)$ formulated for $A(t)$ in various classes of nonlinear operators of montone type (e.g., [3, 4, 11]). Recently, in case $A(t)$ is the subdifferential of a proper lower semicontinuous convex function, various results on the solvability of the evolution equation $(d/dt)u(t) + A(t)u(t) \ni f(t)$ with variable domains have been established (e.g., [1, 5, 7, 10, 12, 13]).

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