49. *Studies on Holonomic Quantum Fields. IV*

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This is a continuation of our previous notes [1], [2] and together with the latter constitutes the second part of the work referred to in [1]. We use the same notation as in [1], [2], [3].

1. First we shall show that the wave function $w_{F,n}={}^t(\hat{w}^1_{F,n}(x), \cdots, \hat{w}^n_{F,n}(x))$ constructed in [2] forms a basis of $W^{\mathrm{strict},R}_{a_1,\cdots,a_n}$. By (30) the local expansion of $w_{F,n}$ in the sense of (10) in [1] takes the form

$$w_{F,n}\sim\frac{i}{2}\Big[\sum_{l=0}^{\infty} C_{F,l}[A]_l w-\sum_{l=0}^{\infty}\bar{C}_{F,l}w^*_l[A]\Big] \tag{42}$$

where $(i/2)C_{F,l}={}^t({}^t\boldsymbol{c}_l(\hat{w}^1_{F,n}), \cdots, {}^t\boldsymbol{c}_l(\hat{w}^n_{F,n}))$. From (31) it follows that if we write $C_{F,0}=1-T$, T is purely imaginary and hermitian: $T=-\bar{T}=-{}^tT$. Since $w_{\mathfrak{R}}$ is a basis of $W^{\mathrm{strict},R}_{a_1,\cdots,a_n}$, there exists a real $n\times n$ matrix C satisfying $w_{F,n}=Cw_{\mathfrak{R}}$. Comparing the 0-th coefficients of their local expansions we have $(i/2)C_{F,0}=CC_{\mathfrak{R},0}$ or equivalently $1-T=2Ce^{-H}$. Taking the complex conjugate we have $1+T=2Ce^{H}$, and hence

$$C=(2\cosh H)^{-1},\qquad T=\tanh H=(1-G)(1+G)^{-1}. \tag{43}$$

Hence $w_{F,n}$ is also a basis of $W^{\mathrm{strict},R}_{a_1,\cdots,a_n}$.

The relation between w_F and $w_{\mathfrak{R}}$ enables us to express the coefficients B, E appearing in the system (12) in [1] satisfied by $w_{\mathfrak{R}}$, in terms of $\tau_{F,n}$ and $\tau^{\mu\nu}_{F,n}$. From (11), (40), (41) and (43) we have

$$\begin{aligned} F&=[U^{-1}V, mA], & G&=U(2\tau_{F,n}-U)^{-1}\\ B&=\sqrt{G}mA\sqrt{G}^{-1}, & E&=\sqrt{G}F\sqrt{G}^{-1}, \end{aligned} \tag{44}$$

where

$$U=\tau_{F,n}(1-T)=\begin{pmatrix} \tau_{F,n} & i\tau^{12}_{F,n} & \cdots & i\tau^{1n}_{F,n}\\ -i\tau^{12}_{F,n} & \tau_{F,n} & \cdots & i\tau^{2n}_{F,n}\\ \vdots & \vdots & \ddots & \vdots\\ -i\tau^{1n}_{F,n} & -i\tau^{2n}_{F,n} & \cdots & \tau_{F,n}\end{pmatrix}$$

$$V=2\begin{bmatrix} m^{-1}\partial_{-a_1^-}\tau_{F,n} & im^{-1}\partial_{-a_2^-}\tau^{12}_{F,n} & \cdots & im^{-1}\partial_{-a_n^-}\tau^{1n}_{F,n}\\ -im^{-1}\partial_{-a_1^-}\tau^{12}_{F,n} & m^{-1}\partial_{-a_2^-}\tau_{F,n} & \cdots & im^{-1}\partial_{-a_n^-}\tau^{2n}_{F,n}\\ \vdots & \vdots & \ddots & \vdots\\ -im^{-1}\partial_{-a_1^-}\tau^{1n}_{F,n} & -im^{-1}\partial_{-a_2^-}\tau^{2n}_{F,n} & \cdots & m^{-1}\partial_{-a_n^-}\tau_{F,n}\end{bmatrix}. \tag{45}$$

Thus we have constructed, in terms of ψ, φ_F and φ^F, not only a solution to the extended holonomic system (12) but also one to the system of total differtntial equations (18).

2. Now we will give a closed expression for $\tau_{F,n}$ by means of solution matrices to the total differential equations (18) in [1]. From (40) and (41) we see that

$$\omega \underset{\text{def}}{=} d \log \tau_{F,n} = \frac{1}{2}(\operatorname{tr} C_{F,1} m dA + \operatorname{tr} \bar{C}_{F,1} m d\bar{A}). \tag{46}$$

From (13) in [1] and (43), after a little computation we rewrite (46) in the following form.

$$\omega = \frac{1}{2} \operatorname{tr}\left[\frac{1}{2}T\Theta - \frac{1}{2}F\Theta + m^2(-{}^tG\bar{A}G + \bar{A})dA\right] + \text{complex conjugate}. \tag{47}$$

We note that the 1-form in the right hand side of (47) is shown to be a closed 1-form and is invariant under the Euclidean motion group even for an arbitrary solution to (18) in [1].

$\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m}(x)$ is written as a linear combination of the components of $w_{F,n}$ as follows.

$$\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m}(x) = \text{Pfaffian}\begin{pmatrix} 0 & \hat{w}_{F,n}^{\nu_1}(x) & \cdots\cdots & \hat{w}_{F,n}^{\nu_m}(x) \\ -\hat{w}_{F,n}^{\nu_1}(x) & 0 & \hat{\tau}_{F,n}^{\nu_1\nu_2} \quad \cdots & \hat{\tau}_{F,n}^{\nu_1,\nu_m} \\ \vdots & \hat{\tau}_{F,n}^{\nu_2\nu_1} & 0 & \vdots \\ \vdots & \vdots & & \hat{\tau}_{F,n}^{\nu_{m-1}\nu_m} \\ -\hat{w}_{F,n}^{\nu_m}(x) & \hat{\tau}_{F,n}^{\nu_m\nu_1} & \cdots\cdots \hat{\tau}_{F,n}^{\nu_m,m-1} & 0 \end{pmatrix}. \tag{48}$$

Comparing the local expansion of both sides of (48) we have

$$\begin{aligned} \hat{\tau}_{F,n}^{\nu_1,\cdots,\nu_m} &= \text{Pfaffian}\begin{pmatrix} 0 & \hat{\tau}_{F,n}^{\nu_1\nu_2} & \cdots\cdots & \hat{\tau}_{F,n}^{\nu_1\nu_m} \\ \hat{\tau}_{F,n}^{\nu_2\nu_1} & 0 & & \vdots \\ \vdots & & & \hat{\tau}_{F,n}^{\nu_{m-1}\nu_m} \\ \hat{\tau}_{F,n}^{\nu_m\nu_1} & \cdots\cdots & \hat{\tau}_{F,n}^{\nu_m\nu_{m-1}} & 0 \end{pmatrix} \\ &= \text{Pfaffian}\,(i(\tanh H)_{\nu,\nu'})_{\nu,\nu'=\nu_1,\cdots,\nu_m}. \end{aligned} \tag{49}$$

More generally we have

$$\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k) = \text{Pfaffian} \tag{50}$$

$$\begin{pmatrix} 0 & \hat{w}_{F,n}(x_1x_2) & \cdots & \hat{w}_{F,n}(x_1x_k) & \hat{w}_{F,n}^{\nu_1}(x_1) & & \hat{w}_{F,n}^{\nu_m}(x_1) \\ -\hat{w}_{F,n}(x_1x_2) & 0 & \cdots & \hat{w}_{F,n}(x_2x_k) & \hat{w}_{F,n}^{\nu_1}(x_2) & & \hat{w}_{F,n}^{\nu_m}(x_2) \\ -\hat{w}_{F,n}(x_1x_k) & -\hat{w}_{F,n}(x_2x_k) & \cdots & 0 & \hat{w}_{F,n}^{\nu_1}(x_k) & & \hat{w}_{F,n}^{\nu_m}(x_k) \\ -\hat{w}_{F,n}^{\nu_1}(x_1) & -\hat{w}_{F,n}^{\nu_1}(x_2) & \cdots & -\hat{w}_{F,n}^{\nu_1}(x_k) & 0 & \hat{\tau}_{F,n}^{\nu_1\nu_2} & \hat{\tau}_{F,n}^{\nu_1\nu_m} \\ & & & & \hat{\tau}_{F,n}^{\nu_2\nu_1} & 0 & \\ & & & & & & \hat{\tau}_{F,n}^{\nu_{m-1}\nu_m} \\ -\hat{w}_{F,n}^{\nu_m}(x_1) & -\hat{w}_{F,n}^{\nu_m}(x_2) & \cdots & -\hat{w}_{F,n}^{\nu_m}(x_k) & \hat{\tau}_{F,n}^{\nu_m\nu_1} & \hat{\tau}_{F,n}^{\nu_m\nu_{m-1}} & 0 \end{pmatrix}.$$

Erratum in Sato-Miwa-Jimbo [3]. The expressions in paragraphs §3 and §4 should be corrected as follows.

p. 7, line 5 from the bottom:

$$\langle w, w' \rangle = \frac{1}{2}\int_{-\infty}^{+\infty} m dx^1 (w_+(x)w'_+(x) + w_-(x)w'_-(x)),$$

lines 4–3 from the bottom:

$$\frac{1}{2}\int_{-\infty}^{+\infty} m dx^1(w_+(x)\psi_+(x)+w_-(x)\psi_-(x)).$$

p. 8, line 13 from the bottom:

$$\phi_\pm(u)=\varepsilon(u)\lim_{t\to\pm\infty}\frac{i}{2}\int_{x^0=t} dx^1(e^{im(x^- u+x^+u^{-1})}(\partial/\partial x^0)\varphi^F(x)$$
$$-\varphi^F(x)(\partial/\partial x^0)e^{im(x^- u+x^+u^{-1})}).$$

References

[1] M. Sato, T. Miwa, and M. Jimbo: Proc. Japan Acad., **53**A, 147–152 (1977).
[2] ——: ibid., 153–158 (1977).
[3] ——: ibid., 6–10 (1977).