# 44. Nonlinear Evolution Equations with Variable Domains in Hilbert Spaces

By Nobuyuki KENMOCHI
Department of Mathematics, Faculty of Education, Chiba University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1977)

Let $H$ be a real Hilbert space and denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $H$, respectively. Let $\phi^t$ be a proper lower semi-continuous convex function on $H$ and put $D_t=\{v \in H\,;\phi^t(v)<+\infty\}$ and $D(\partial\phi^t)=\{v \in H\,;\partial\phi^t(v)\neq\emptyset\}$ for each $t \in [0,T]$, where $0<T<+\infty$ and $\partial\phi^t$ is the subdifferential of $\phi^t$. In this paper we consider the evolution equation

(E) $\qquad u'(t)+\partial\phi^t(u(t)) \ni f(t), \qquad t \in [0,T],$

where $u'(t)=(d/dt)u(t)$ and $f$ is given in $L^2(0,T\,;H)$.

In recent years the evolution equation (E) with time-dependent domain $D(\partial\phi^t)$ has been studied by Attouch-Bénilan-Damlamian-Picard [1], Brézis [3], Moreau [7], Kenmochi [5] and Yamada [11]. In the same direction we further study the equation (E).

For each $\lambda>0$ and $t \in [0,T]$, define

$$\phi^t_\lambda(v)=\inf\{\|v-z\|^2/(2\lambda)+\phi^t(z)\,;z \in H\}, \qquad v \in H.$$

According to [4; Chap. II], we see that

$$\partial\phi^t_\lambda(v)=(v-J^t_\lambda v)/\lambda$$

and

$$\phi^t_\lambda(v)=\|v-J^t_\lambda v\|^2/(2\lambda)+\phi^t(J^t_\lambda v)$$

for each $v \in H$, where $J^t_\lambda=(I+\lambda\partial\phi^t)^{-1}$.

Now suppose that

(h1) *there are positive constants $\alpha$ and $\beta$ such that $\phi^t(z)+\alpha\|z\|+\beta \geqq 0$ for any $t \in [0,T]$ and $z \in H$;*

(h2) *for each $\lambda>0$ and $z \in H$ there is a non-negative function $\rho \in L^1(0,T)$ such that*

$$\phi^t_\lambda(z)-\phi^s_\lambda(z) \leqq \int_s^t \rho(\tau)d\tau$$

*for $s,t \in [0,T]$ with $s \leqq t$;*

(h3) (i) *for each $r \geqq 0$, there are a number $a_r \in [0,1)$ and functions $b_r, c_r \in L^1(0,T)$ such that $(d/dt)\phi^t_\lambda(z) \leqq a_r\|\partial\phi^t_\lambda(z)\|^2+b_r(t)|\phi^t_\lambda(z)|+c_r(t)$* a.e. *on $[0,T]$ for $z \in H$ with $\|z\| \leqq r$ and $\lambda \in (0,1]$; and* (ii) *there are an $H$-valued function $h$ on $[0,T]$ and a partition $\{0=t_0<t_1<\cdots<t_N=T\}$ of $[0,T]$ such that $\phi^t(h(t)) \in L^1(0,T)$ and the restriction of $h$ to $(t_{k-1},t_k)$ belongs to $W^{1,1}(t_{k-1},t_k\,;H)$ for $k=1,2,\cdots,N$.*

**Theorem.** *For each $u_0 \in \bar{D}_0$ and $f \in L^2(0,T\,;H)$ there exists a* *unique function* $u\in C([0,T];H)$ *satisfying that* $u(0)=u_0$, $\sqrt{t}u'\in L^2(0,T;H)$ *and* $u'(t)+\partial\phi^t(u(t))\ni f(t)$ *for* a.e. $t\in[0,T]$. *Furthermore* $u(t)\in D_t$ *for all* $t\in(0,T]$ *and the function* $t\to t\phi^t(u(t))$ *is bounded on* $(0,T]$. *In particular, if* $u_0\in D_0$, *then* $u'\in L^2(0,T;H)$ *and* $t\to\phi^t(u(t))$ *is bounded on* $[0,T]$.

This theorem is able to be obtained in a way quite similar to that in [1] (for details, see [9]).

**Remark 1.** When (h3) is replaced by the following (h3)′, the same conclusion in the theorem remains valid:

(h3)′ *There are a number* $a\in[0,1)$ *and functions* $b,c\in L^1(0,T)$ *such that*

$$(d/dt)\phi^t_\lambda(z)\leqq a\,\|\partial\phi^t_\lambda(z)\|^2+b(t)\,|\phi^t_\lambda(z)|+(1+\|z\|^2)c(t)\qquad\text{a.e. on } [0,T]$$

*for every* $z\in H$ *and* $\lambda\in(0,1]$; *in this case we do not require* (ii) *of* (h3).

The following proposition gives a useful condition under which (h1), (h2) and (h3) hold.

**Proposition.** *Suppose that for each* $r\geqq 0$ *there are real-valued functions* $\alpha_r\in W^{1,2}(0,T)$ *and* $\beta_r\in W^{1,1}(0,T)$ *with the following property: for each* $s,t\in[0,T]$ *with* $s\leqq t$ *and* $v\in D_s$ *with* $\|v\|\leqq r$ *there exists* $w\in D_t$ *such that*

$$\|w-v\|\leqq|\alpha_r(t)-\alpha_r(s)|(1+|\phi^s(v)|^{1/2})$$

*and*

$$\phi^t(w)-\phi^s(v)\leqq|\beta_r(t)-\beta_r(s)|(1+|\phi^s(v)|).$$

*Then* (h1), (h2) *and* (h3) *are satisfied.*

First, we refer to [2; Lemma 1] (or [5; Lemma 3.2]) for the verification of (h1). Next, we note that for each $r\geqq 0$ there is $r_1\geqq 0$ such that $\|J^t_\lambda z\|\leqq r_1$ for all $t\in[0,T]$, $\lambda\in(0,1]$ and $z\in H$ with $\|z\|\leqq r$. Let $z\in H$ with $\|z\|\leqq r$ and $\lambda\in(0,1]$. Then for $s,t\in[0,T]$ with $s\leqq t$, we find by assumption $w\in D_t$ so that

$$\|w-J^s_\lambda z\|\leqq|\alpha_{r_1}(t)-\alpha_{r_1}(s)|(1+|\phi^s_\lambda(z)|^{1/2})$$

and

$$\phi^t(w)-\phi^s(J^s_\lambda z)\leqq|\beta_{r_1}(t)-\beta_{r_1}(s)|(1+|\phi^s_\lambda(z)|).$$

Hence

$$\begin{aligned}
&\phi^t_\lambda(z)-\phi^s_\lambda(z)\\
&\quad\leqq\|z-w\|^2/(2\lambda)+\phi^t(w)-\|z-J^s_\lambda z\|^2/(2\lambda)-\phi^s(J^s_\lambda z)\\
&\quad\leqq\|w-J^s_\lambda z\|\cdot\|z-J^s_\lambda z\|/\lambda+\phi^t(w)-\phi^s(J^s_\lambda z)+\|w-J^s_\lambda z\|^2/(2\lambda)\\
&\quad\leqq|\alpha_{r_1}(t)-\alpha_{r_1}(s)|\cdot\|\partial\phi^s_\lambda(z)\|(1+|\phi^s_\lambda(z)|^{1/2})+|\beta_{r_1}(t)-\beta_{r_1}(s)|(1+|\phi^s_\lambda(z)|)\\
&\qquad+|\alpha_{r_1}(t)-\alpha_{r_1}(s)|^2(1+|\phi^s_\lambda(z)|^{1/2})^2/(2\lambda),
\end{aligned}$$

so that $(d/ds)\phi^s_\lambda(z)\leqq|\alpha'_{r_1}(s)|\cdot\|\partial\phi^s_\lambda(z)\|(1+|\phi^s_\lambda(z)|^{1/2})+|\beta'_{r_1}(s)|(1+|\phi^s_\lambda(z)|)$ for a.e. $s\in[0,T]$. Thus (i) of (h3) is satisfied with (h2). To verify (ii) of (h3) we observe that there are $R>0$ and a set $\{z_t\in D_t;\,0\leqq t\leqq T\}$ such that $\|z_t\|\leqq R$ and $|\phi^t(z_t)|\leqq R$ for all $t\in[0,T]$. Now, take $r>R+1$, put $M=R+\alpha r+\beta+1$ ($\alpha$ and $\beta$ are constants such as in (h1)) and choose $\eta>0$

so that

$$\left\{1+M\exp\left(\int_0^T|\beta_r'|\,d\tau\right)\right\}\int_{I(t)}|\alpha_r'|\,d\tau\leqq 1$$

for all $t\in[0,T]$, where $I(t)=[t,t(\eta)]$ with $t(\eta)=\min\{t+\eta,T\}$. Then for each $t\in[0,T]$ there is $h_t\in W^{1,2}(\mathring{I}(t);H)$ satisfying that $s\to\phi^s(h_t(s))$ is bounded on $I_t$; in fact, for each partition $\Delta_n=\{t=s_0^n<s_1^n<\cdots<s_{N(n)}^n=t(\eta)\}$ with $s_k^n=t+k|\Delta_n|$ and $|\Delta_n|=(t(\eta)-t)/2^n$, we can build by induction a sequence $\{v_k^n\}$ such that $v_0^n=z_t$, $\|v_k^n\|\leqq r$,

$$\|v_{k+1}^n-v_k^n\|\leqq\left\{1+M\exp\left(\int_0^T|\beta_r'|\,d\tau\right)\right\}\int_{s_k^n}^{s_{k+1}^n}|\alpha_r'|\,d\tau$$

and

$$\phi^{s_{k+1}^n}(v_{k+1}^n)\leqq\phi^{s_k^n}(v_k^n)+M\exp\left(\int_0^T|\beta_r'|\,d\tau\right)\int_{s_k^n}^{s_{k+1}^n}|\alpha_r'|\,d\tau$$

for $k=0,1,\cdots,N(n)-1$. Besides, putting $v_n(s)=v_k^n$ and $\nabla_n v_n(s)=(v_k^n-v_{k+1}^n)/|\Delta_n|$ for $s\in(s_k^n,s_{k+1}^n]$, we are able to show that suitable subsequences of $\{v_n\}$ and $\{\nabla_n v_n\}$ converge weakly to some functions $h_t$ and $\bar{h}_t$ in $L^2(\mathring{I}(t);H)$, respectively, and that $s\to\phi^s(h_t(s))$ is bounded on $I(t)$. Since $\bar{h}_t=h_t'$ clearly, this function $h_t$ is the desired one. Making use of the family $\{h_t;0\leqq t\leqq T\}$ we easily obtain an $H$-valued function $h$ and a partition of $[0,T]$ required in (ii) of (h3).

**Remark 2.** Our hypothesis in the proposition seems to be checked more easily than that imposed by Yamada [11]. Also, compare the hypotheses by Watanabe [10], Péralba [8], Attouch-Damlamian [2], Maruo [6] and Kenmochi [5] with ours.

**Remark 3.** The above results were suggested by H. Brézis.

## References

[1] H. Attouch, Ph. Bénilan, A. Damlamian, and C. Picard: Equations d'évolution avec condition unilatérale. C. R. Acad. Sci. Paris, **279**, 607–609 (1974).
[2] H. Attouch and A. Damlamian: Problèmes d'évolution dans les Hilbert et applications. J. Math. Pures et Appl., **54**, 53–74 (1975).
[3] H. Brézis: Un problème d'évolution avec contraintes unilatérales dépendant du temps. C. R. Acad. Sci. Paris, **274**, 310–312 (1972).
[4] ——: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland, 1973.
[5] N. Kenmochi: Some nonlinear parabolic variational inequalities. Israel J. Math., **22**, 304–331 (1975).
[6] K. Maruo: On some evolution equations of subdifferential operators. Proc. Japan Acad., **51**, 304–307 (1975).
[7] J. J. Moreau: Problème d'évolution associé à un convexe mobile d'un espace hilbertien. C. R. Acad. Sci. Paris, **276**, 791–794 (1973).
[8] J. C. Péralba: Un problème d'évolution relatif à un opérateur sous-différentiel dépendant du temps. C. R. Acad. Sci. Paris, **275**, 93–96 (1972).
[9] C. Picard: Equations d'évolution avec condition unilatérale. Séminaire Lions, 1973/74.

[10] J. Watanabe: On certain nonlinear evolution equations. J. Math. Soc. Japan, **25**, 446–463 (1973).
[11] Y. Yamada: On evolution equations generated by subdifferential operators. J. Fac. Sci. Univ. Tokyo, **23**, 491–515 (1976).