

41. Studies on Holonomic Quantum Fields. II

By Mikio SATO, Tetsuji MIWA, and Michio JIMBO

Research Institute of Mathematical Sciences, Kyoto University

(Communicated by Kôzaku YOSIDA, M. J. A., Oct. 12, 1977)

This is a continuation of our preceding work [1] on the construction and study of models of holonomic quantum fields. The present work consists of two parts and is presented in three separate notes accordingly.

The first part included here is a mathematical preliminary concerning deformation of a holonomic system, which seems to be of its own interest. We consider the space $W_{a_1, \dots, a_n}^{\text{strict}}$ of double-valued solutions to the 2-dimensional Euclidean Dirac equation satisfying suitable growth order conditions at the branch points a_1, \dots, a_n and at ∞ . After establishing its finite dimensionality, we derive a holonomic system of first order linear differential equations satisfied by a basis of $W_{a_1, \dots, a_n}^{\text{strict}}$. The coefficients appearing in these equations are functions of a_1, \dots, a_n and are shown to satisfy a completely integrable system of total differential equations.

Applying these results, we show in the coming second part that the n -point τ -functions of the fields φ_F and φ^F constructed in [1] are expressible in terms of the solutions to the total differential equations derived in II-§4. This generalizes the remarkable result of [2] which says that the 2-point functions of φ_F and φ^F have closed expressions in terms of the Painlevé function of the third kind. Also we derive some algebraic relations between various vacuum expectation values of products of the fields ψ , φ_F and φ^F .

In this note we deal with the Euclidean 2-space $R^2 = \{(x^1, x^2)\}$. We use the coordinate $z = \frac{x^1 + ix^2}{2}$, $\bar{z} = \frac{x^1 - ix^2}{2}$ and set $\partial_z = \partial/\partial z$, $\partial_{\bar{z}} = \partial/\partial \bar{z}$.

1. Consider the 2-dimensional Euclidean Dirac equation with positive mass m

$$(1) \quad (m - \Gamma)w = 0, \quad \Gamma = \begin{pmatrix} & \partial_z \\ \partial_{\bar{z}} & \end{pmatrix}, \quad w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix}.$$

Denote by \mathcal{D} the sheaf of 2×2 matrices of differential operators, and set $\mathcal{D}_0 = \{P \in \mathcal{D} \mid \mathcal{G} \cdot P \subset \mathcal{G}\}$ where $\mathcal{G} = \mathcal{D}(m - \Gamma)$. Then \mathcal{D}_0 is the unique maximal subring of \mathcal{D} containing \mathcal{G} as its bi-ideal, and if w satisfies (1), so does Pw for $P \in \mathcal{D}_0$. We have $\mathcal{D}_0 = \mathcal{G} + C[\partial_z, \partial_{\bar{z}}, M_F]$, where $\partial_z, \partial_{\bar{z}}$ and $M_F = z\partial_z - \bar{z}\partial_{\bar{z}} + \frac{1}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ are the infinitesimal generators of the

Euclidean motion group. A solution w to (1) is called a wave function, and it is said to be real if $w^*=w$, where the $*$ -conjugation is defined by $w^* = \begin{pmatrix} \bar{w}_- \\ \bar{w}_+ \end{pmatrix}$. For $l \in \mathbb{C}$ we set

$$(2) \quad w_l(z, \bar{z}) = \begin{pmatrix} v_{l-1/2}(z, \bar{z}) \\ v_{l+1/2}(z, \bar{z}) \end{pmatrix}, \quad w_l^*(z, \bar{z}) = \begin{pmatrix} v_{l+1/2}^*(z, \bar{z}) \\ v_{l-1/2}^*(z, \bar{z}) \end{pmatrix}$$

where $v_l, v_l^* = e^{\pm i l \theta} I_l(mr)$, $z, \bar{z} = (1/2)re^{\pm i \theta}$ ($r \geq 0$) and $I_l(mr)$ denotes the modified Bessel function of the 1st kind. These are solutions to (1) with the properties $\partial_z w_l = m w_{l-1}$, $\partial_z w_l^* = m w_{l+1}^*$, $\partial_{\bar{z}} w_l = m w_{l+1}$, $\partial_{\bar{z}} w_l^* = m w_{l-1}^*$, $M_F w_l = l w_l$ and $M_F w_l^* = -l w_l^*$. Moreover any (multi-valued) local solution w at $(z, \bar{z}) = (a, \bar{a})$ such that $w(a + e^{2\pi i}(z-a), \bar{a} + e^{-2\pi i}(\bar{z}-\bar{a})) = e^{2\pi i(l_0+1/2)} w(z, \bar{z})$ and $|w| = O(|z-a|^{\text{Re } l_0-1/2})$ ($|z-a| \rightarrow 0$) for some $l_0 \in \mathbb{C}$ is uniquely expanded in the form

$$(3) \quad w = \sum_{l \equiv l_0 \pmod{\mathbb{Z}}} c_l w_l[a] + \sum_{l \equiv -l_0 \pmod{\mathbb{Z}}} c_l^* w_l^*[a], \quad c_l, c_l^* \in \mathbb{C}, \text{Re } l \geq \text{Re } l_0$$

if $l_0 \not\equiv 1/2 \pmod{\mathbb{Z}}$, where $w_l[a] = w_l(z-a, \bar{z}-\bar{a})$ and $w_l^*[a] = w_l^*(z-a, \bar{z}-\bar{a})$. In the case $l_0 \equiv 1/2 \pmod{\mathbb{Z}}$, the expansion takes the form

$$(4) \quad w = \sum_{\substack{l \equiv 1/2 \pmod{\mathbb{Z}} \\ l \geq \max(1/2, l_0)}} (c_l w_l[a] + c_l^* w_l^*[a]) + \sum_{\substack{l \equiv 1/2 \pmod{\mathbb{Z}} \\ -l_0 \geq l \geq 1/2}} (\tilde{c}_l \tilde{w}_l[a] + \tilde{c}_l^* \tilde{w}_l^*[a]),$$

where \tilde{w}_l and \tilde{w}_l^* are constructed from $\tilde{v}_l, \tilde{v}_l^* = e^{\pm i l(\theta + \pi)} K_l(mr)$ in a similar way, and if $l_0 = 1/2$ we replace the assumption $|w| = O(|z-a|^{\text{Re } l_0-1/2})$ by $|w| = O(\log |z-a|)$. We say w is of Fermi-type at (a, \bar{a}) if $l_0 \in \mathbb{Z}$, and of strict Fermi-type if further $l_0 \geq 0$.

2. Let $W_{a_1, \dots, a_n} / W_{a_1, \dots, a_n}^{\text{strict}}$ denote the set of wave functions which are of Fermi-/strict Fermi-type at (a_μ, \bar{a}_μ) ($\mu=1, \dots, n$) and which behave like $O(e^{-2m|z|})$ as $|z| \rightarrow \infty$. More precisely an element $w \in W_{a_1, \dots, a_n}$ is a function defined on the 2-fold ramified covering manifold $\mathcal{R} = \left\{ (z, \bar{z}, \zeta, \bar{\zeta}) \mid \zeta^2 - \prod_{\mu=1}^n (z-a_\mu) = 0, \bar{\zeta}^2 - \prod_{\mu=1}^n (\bar{z}-\bar{a}_\mu) = 0 \right\}$ of \mathbb{R}^2 , which is odd i.e. changes sign under $(z, \bar{z}, \zeta, \bar{\zeta}) \mapsto (z, \bar{z}, -\zeta, -\bar{\zeta})$ and which satisfies (1) outside the branch points $(a_\mu, \bar{a}_\mu, 0, 0) \in \mathcal{R}$ ($\mu=1, \dots, n$). Since $\partial_z, \partial_{\bar{z}}$ and M_F belong to \mathcal{D}_0 , W_{a_1, \dots, a_n} is a left $\mathcal{C}[\partial_z, \partial_{\bar{z}}, M_F]$ -module. For $w \in W_{a_1, \dots, a_n}$ we denote by $c_l^{(\mu)}(w)$ (resp. $c_l^{(\mu)*}(w)$) the coefficient of $w_l[a_\mu]$ (resp. $w_l^*[a_\mu]$) in the local expansion (3) at (a_μ, \bar{a}_μ) . We denote by $W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}}$ the set of real elements in $W_{a_1, \dots, a_n}^{\text{strict}}$. Taking a local parameter $\zeta_\mu = \sqrt{z-a_\mu} = i\xi_\mu^0 + \xi_\mu^1$, the "strict Fermi" condition is equivalent to the Dirac equation with point sources

$$\begin{pmatrix} 2m\zeta_\mu & -\partial_{\zeta_\mu} \\ -\partial_{\zeta_\mu} & 2m\bar{\zeta}_\mu \end{pmatrix} w = -\sqrt{\frac{\pi}{m}} \begin{pmatrix} c_0^{(\mu)*}(w) \\ c_0^{(\mu)}(w) \end{pmatrix} \delta(\xi_\mu^0) \delta(\xi_\mu^1) \quad (\mu=1, \dots, n).$$

For $w, w' \in W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}}$ we set

$$(5) \quad I(w, w') = \frac{1}{4} \text{Re} \iint_{\mathcal{R} - \cup_{\mu=1}^n V_\mu^0} im^2 dz d\bar{z} (w_+ \bar{w}'_+ + w_- \bar{w}'_-) + \frac{1}{4} \sum_{\mu=1}^n \lim_{\epsilon \downarrow 0} \text{Re} \iint_{V_\mu^\epsilon} 4im^2 \zeta_\mu \bar{\zeta}_\mu d\zeta_\mu d\bar{\zeta}_\mu (w_+ \bar{w}'_+ + w_- \bar{w}'_-)$$

where $V_\mu^\varepsilon = \{\varepsilon \leq |\zeta_\mu| \leq r\}$ with sufficiently small $r > \varepsilon \geq 0$. Then $I(\cdot)$ defines a positive definite symmetric inner product on $W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}}$. In terms of $c_i(w) = (c_i^{(1)}(w), \dots, c_i^{(n)}(w))$ it is given by

$$I(w, w') = \frac{1}{2} \lim_{\varepsilon \downarrow 0} \sum_{\mu=1}^n \left(\oint_{|\zeta_\mu|=\varepsilon} m i \zeta_\mu d\zeta_\mu w_+ w'_+ \right) = -c_0(w)^t c_0(w').$$

Consequently the map $W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}} \ni w \mapsto c_0(w) \in \mathbb{C}^n$ is injective, and since its image does not contain isotropic elements, we have $\dim_{\mathbb{R}} W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}} \leq n$. Later in III-§2 we shall construct explicitly n independent elements of $W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}}$ using our operators ψ, φ_F and φ^F [1]. Existence of such a basis is also shown by the following argument. First by the method of orthogonal projection we construct on \mathcal{R} solutions v_μ, \bar{v}_μ ($\mu=1, \dots, n$) to the Euclidean Klein-Gordon equation $(m^2 - \partial_z \partial_{\bar{z}})v = 0$ with the properties $v_\mu, \bar{v}_\mu = O(e^{-2m|z|})$ ($|z| \rightarrow \infty$) and

$$(6) \quad \begin{aligned} v_\mu &= \delta_{\mu\nu} v_{-1/2}[a_\nu] + \alpha_{\mu\nu} v_{1/2}[a_\nu] + \beta_{\mu\nu} v_{1/2}^*[a_\nu] + \dots \\ \bar{v}_\mu &= \delta_{\mu\nu} v_{-1/2}^*[a_\nu] + \bar{\beta}_{\mu\nu} v_{1/2}[a_\nu] + \bar{\alpha}_{\mu\nu} v_{1/2}^*[a_\nu] + \dots \end{aligned}$$

at (a_ν, \bar{a}_ν) ($\mu, \nu=1, \dots, n$). Setting $\alpha = (\alpha_{\mu\nu})$ and $\beta = (\beta_{\mu\nu})$ we find that $\alpha = {}^t \alpha$ and that $\beta = {}^t \beta^{-1} = \bar{\beta}^{-1}$ is negative definite, in other words $-\beta = e^{2H}$ with a unique $H = -{}^t H = -\bar{H}$. Let $K = (k_{\mu\nu})$ be a non-singular matrix satisfying $K\beta = \bar{K}$, e.g. $K = ie^{-H}$. Then $\left\{ w_\mu = \begin{pmatrix} w_\mu^+ \\ w_\mu^- \end{pmatrix} \right\}_{\mu=1, \dots, n}$ defined by $w_\mu^+ = \overline{w_\mu^-}$

$= \sum_{\nu=1}^n k_{\mu\nu} v_\nu$ provides a basis of $W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}}$.

There is a natural map $(\mathbb{C}[\partial_z, \partial_{\bar{z}}]/(m^2 - \partial_z \partial_{\bar{z}})) \otimes_{\mathbb{C}} W_{a_1, \dots, a_n}^{\text{strict}} \rightarrow W_{a_1, \dots, a_n}$

defined by $p(\partial_z, \partial_{\bar{z}}) \otimes w \mapsto p(\partial_z, \partial_{\bar{z}})w$, which is clearly injective. Let us show that it is also surjective. For any $w \in W_{a_1, \dots, a_n}$ let $c_{l_0}(w), c_{l_0}^*(w)$ be its coefficient vectors of the first term in the local expansion (3). Since K is non-singular, there exist row vectors $k, k^* \in \mathbb{C}^n$ such that $c_{l_0}(w) = k \cdot K, c_{l_0}^*(w) = k^* \cdot \bar{K}$. Then

$$w^{(1)} = w - \sum_{\mu=1}^n (k_\mu (m^{-1} \partial_z)^{-l_0} w_\mu + k_\mu^* (m^{-1} \partial_z)^{-l_0} w_\mu^*)$$

satisfies $c_l(w^{(1)}) = 0, c_l^*(w^{(1)}) = 0$ for $l \leq l_0$. Continuing this process $(-l_0 + 1)$ -times we have $w^{(-l_0+1)} \in W_{a_1, \dots, a_n}^{\text{strict}}$ and $c_0(w^{(-l_0+1)}) = 0, c_0^*(w^{(-l_0+1)}) = 0$, which imply $w^{(-l_0+1)} = 0$. Thus we have

$$(7) \quad (\mathbb{C}[\partial_z, \partial_{\bar{z}}]/(m^2 - \partial_z \partial_{\bar{z}})) \otimes_{\mathbb{C}} W_{a_1, \dots, a_n}^{\text{strict}} \xrightarrow{\sim} W_{a_1, \dots, a_n}.$$

3. Let $\{w_\mu\}_{\mu=1, \dots, n}$ be a basis of $W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}}$ and set $\mathbf{w} = {}^t(w_1, \dots, w_n)$. Then for each $\nu=1, \dots, n, M_F w_\nu \in W_{a_1, \dots, a_n} = \mathbb{C}[\partial_z, \partial_{\bar{z}}] W_{a_1, \dots, a_n}^{\text{strict}, \mathbb{R}}$, whence it follows from (7) that there exist unique constant $n \times n$ matrices B, E with $\bar{E} = -E$ so that

$$(8) \quad M_F \mathbf{w} = (B m^{-1} \partial_z - \bar{B} m^{-1} \partial_{\bar{z}} + E) \mathbf{w}$$

holds. Equation (8) together with the Dirac equation (1) constitutes the holonomic system for \mathbf{w} . Denote by $C_l = (c_i^{(l)}(w_\mu))_{\mu, \nu=1, \dots, n}$ the matrix of the l -th coefficients in the local expansion (3). By introducing $C(\partial_z)$

$= \sum_{l=0}^{\infty} C_l(m^{-1}\partial_z)^l$ we write as

$$(9) \quad w \sim C(\partial_z)w_0[A] + \bar{C}(\partial_z)w_0^*[A] = \sum_{l=0}^{\infty} C_l w_l[A] + \sum_{l=0}^{\infty} \bar{C}_l w_l^*[A],$$

where $A = (\delta_{\mu\nu} \alpha_\nu)$ and $w_l[A]$ (resp. $w_l^*[A]$) denotes the diagonal matrix $(\delta_{\mu\nu} w_l[a_\nu])$ (resp. $(\delta_{\mu\nu} w_l^*[a_\nu])$). In terms of $C(\partial_z)$ equation (8) reduces to $PC(\partial_z) = C(\partial_z)P_0$ where we have set $P = M_F - Bm^{-1}\partial_z + \bar{B}m^{-1}\partial_z - E$, $P_0 = M_F - A\partial_z + \bar{A}\partial_z$, or equivalently, to

$$(10) \quad BC_l - C_l mA = (l-1-E)C_{l-1} + \bar{B}C_{l-2} - C_{l-2}m\bar{A},$$

with $C_l = 0$ for $l < 0$. In particular for $l = 0, 1, 2$ we have

$$(11) \quad \begin{aligned} C_0^{-1}BC_0 &= mA, \quad C_0^{-1}EC_0 = [C_0^{-1}C_1, mA], \\ C_0^{-1}EC_1 - C_0^{-1}C_1 + m\bar{A} - C_0^{-1}\bar{C}_0m\bar{A}C_0^{-1}C_0 &= [C_0^{-1}C_2, mA]. \end{aligned}$$

From the argument in §2, there exists a unique basis $w_{\mathcal{R}}$ for which the operator $C(\partial_z) = C_{\mathcal{R}}(\partial_z)$ satisfies $C_{\mathcal{R},0} = i\sqrt{-\beta}^{-1} = ie^{-H}$ and $C_{\mathcal{R},0}^{-1}C_{\mathcal{R},1} = \alpha = {}^t(C_{\mathcal{R},0}^{-1}C_{\mathcal{R},1})$. In this case ${}^tB = B$ and ${}^tE = -E$, as seen from (11).

Consider in general a holonomic system of the form

$$(12) \quad (m - \Gamma)w = 0, \quad (M_F - Bm^{-1}\partial_z + \bar{B}m^{-1}\partial_z - E)w = 0,$$

where $w = {}^t(w_1, \dots, w_n)$, ${}^tB = B$, ${}^tE = -E = \bar{E}$ and the eigenvalues $\{m\alpha_\mu\}_{\mu=1, \dots, n}$ of B are assumed to be distinct. The characteristic variety of (12) is confined to the union of conormal bundles of $z - \alpha_\mu = 0$ or $\bar{z} - \bar{\alpha}_\mu = 0$ ($\mu = 1, \dots, n$). At $(z, \bar{z}) \in \mathbb{R}^2 - \{(a_\mu, \bar{a}_\mu)\}_{\mu=1, \dots, n}$ (12) admits $2n$ independent real regular local solutions, which are analytically continued to the universal covering manifold of $\mathbb{R}^2 - \{(a_\mu, \bar{a}_\mu)\}_{\mu=1, \dots, n}$. At $(z, \bar{z}) = (a_\mu, \bar{a}_\mu)$ there exist $2(n-1)$ independent regular solutions and two of strict Fermi-type. This means that the monodromy representation on the space $C^{\mathcal{V}^R}$ of real solutions defined on the universal covering manifold

$$(13) \quad \pi_1(\mathbb{R}^2 - \{(a_\mu, \bar{a}_\mu)\}_{\mu=1, \dots, n}) \rightarrow GL(C^{\mathcal{V}^R}), \quad \gamma \mapsto C_\gamma$$

takes the form $C_{\gamma_\mu} = U_\mu^{-1} \begin{pmatrix} -I_2 & \\ & I_{2n-2} \end{pmatrix} U_\mu$ for some $U_\mu \in GL(2n, \mathbb{R})$, where γ_μ denotes a cycle encircling (a_μ, \bar{a}_μ) in the positive direction.

The case corresponding to $w_{\mathcal{R}}$ is of special character; namely, the monodromy representation (13) is reducible, and $C^{\mathcal{V}^R}$ contains one dimensional subspace whose generator is of strict Fermi-type at every (a_μ, \bar{a}_μ) ($\mu = 1, \dots, n$) and decreases exponentially at ∞ .

4. For fixed distinct a_1, \dots, a_n , $w_{\mathcal{R}}$ satisfies a holonomic system (12) with B and E such that $B = e^{-H}mAe^H$, $\bar{H} = {}^tH = -H$ and $\bar{E} = {}^tE = -E$. Now we consider how H and E depend on a_1, \dots, a_n . $\partial_{a_\mu} w_{\mathcal{R}}$ and $\partial_{\bar{a}_\mu} w_{\mathcal{R}}$ ($\mu = 1, \dots, n$) belong to W_{a_1, \dots, a_n} ; hence (7) shows that there exists a unique first order differential operator whose coefficients are $n \times n$ matrices of 1-forms in (A, \bar{A})

$$(14) \quad \Omega = \Phi m^{-1}\partial_z + \bar{\Phi} m^{-1}\partial_z + \Psi$$

so that $d w_{\mathcal{R}} = \Omega w_{\mathcal{R}}$. Here we denote by d the exterior differentiation with respect to (A, \bar{A}) . The local expansion of $d w_{\mathcal{R}}$ takes the form

$-C_{\mathcal{R},0}mdAw_{-1}[A] + (dC_{\mathcal{R},0} - C_{\mathcal{R},1}mdA)w_0[A] + \dots + \text{*conjugates}$.
Hence we have $\Phi = -C_{\mathcal{R},0}mdAC_{\mathcal{R},0}^{-1}$ and $\Psi = dC_{\mathcal{R},0}C_{\mathcal{R},0}^{-1} + [C_{\mathcal{R},1}C_{\mathcal{R},0}^{-1}, \Phi]$.

We now study in general a solution w to (12) whose dependence on parameters is governed by

$$(15) \quad d\mathbf{w} = \Omega\mathbf{w}$$

with Ω given by (14) where Φ and Ψ are supposed to satisfy $[\Phi, B]=0$ and $-{}^t\Psi = \Psi = \bar{\Psi}$. The coupled equations (12)+(15) yield $0 = d(P\mathbf{w}) = dP \cdot \mathbf{w} + P \cdot d\mathbf{w} = (dP - [\Omega, P])\mathbf{w}$, $0 = d(d\mathbf{w}) = (d\Omega - \Omega \wedge \Omega)\mathbf{w}$. We now assume that P and Ω satisfy

$$(16) \quad dP - [\Omega, P] \equiv 0, \quad d\Omega - \Omega \wedge \Omega \equiv 0 \pmod{\mathcal{D}(m^2 - \partial_z \partial_{\bar{z}})},$$

or equivalently, the following completely integrable system of total differential equations for functions of (A, \bar{A}) :

$$(17) \quad \begin{aligned} dB &= -\Phi + [\Psi, B] + [\Phi, E] & d\Phi &= [\Phi, \Psi]_+ \\ d\bar{B} &= -\bar{\Phi} + [\bar{\Psi}, \bar{B}] + [\bar{\Phi}, \bar{E}] & d\bar{\Phi} &= [\bar{\Phi}, \bar{\Psi}]_+ \\ dE &= [\bar{\Phi}, B] - [\Phi, \bar{B}] + [\Psi, E] & d\Psi &= [\Phi, \bar{\Phi}]_+ + \Psi \wedge \bar{\Psi}, \end{aligned}$$

and conclude that they guarantee the consistency of our extended holonomic system (12)+(15) for $w(z, \bar{z}; A, \bar{A})$. Namely if B, E, Φ and Ψ satisfy (17), there exists a system of $2n$ independent solutions $(w^{(1)}, \dots, w^{(2n)})$ to (12) which satisfy $d\mathbf{w}^{(\mu)} = \Omega\mathbf{w}^{(\mu)}$ ($\mu=1, \dots, 2n$). For the case $\mathbf{w} = \mathbf{w}_{\mathcal{R}}$, equation (16) is a consequence of (7) since $dP - [\Omega, P]$ and $d\Omega - \Omega \wedge \Omega$ belong to $C[\partial_z, \partial_{\bar{z}}]$ (or more precisely to $C[\partial_z, \partial_{\bar{z}}] \otimes \{n \times n \text{ matrices of exterior differential forms in } A, \bar{A}\}$).

Let the analytic continuation of $(w^{(1)}, \dots, w^{(2n)})$ along a cycle γ be given by $(w^{(1)}, \dots, w^{(2n)})C_\gamma$. We then have $d[(w^{(1)}, \dots, w^{(2n)})C_\gamma] = \Omega[(w^{(1)}, \dots, w^{(2n)})C_\gamma]$, and hence $dC_\gamma = 0$. This means that the monodromy representation (13) stays constant along each integral manifold of (17). Asymptotic behavior of $w(z, \bar{z}; A, \bar{A})$ at $|z| = \infty$ (on each sheet of the covering space) is also invariant along each integral manifold.

For later convenience we rewrite (17) as follows. First we see that $\Phi = -e^{-H}mdA \cdot e^H$ and $\Psi = de^{-H} \cdot e^H + e^{-H}\Theta e^H$ where Θ is a skew-symmetric matrix of 1-forms characterized by $[A, \Theta] + [dA, e^H E e^{-H}] = 0$. (17) now reduces to

$$(18) \quad \begin{aligned} dF &= -[F, \Theta] + m^2[dA, {}^tG\bar{A}G] + m^2[A, {}^tGd\bar{A} \cdot G] \\ dG &= -G\Theta + \bar{\Theta}G, \quad d\Theta = \Theta \wedge \Theta + m^2[dA, {}^tGd\bar{A} \cdot G]_+ \end{aligned}$$

where $G = e^{-2H}$ and $F = e^H E e^{-H}$. We remark that in the case $n=2$ (18) reduces to the restricted Painlevé equation of the third kind with $\nu=0$ in [3].

References

[1] Sato, M., Miwa, T., and Jimbo, M.: Proc. Japan Acad., 53A, 6-10 (1977).

- [2] Wu, T. T., McCoy, B. M., Tracy, C. A., and Barouch, E.: Phys. Rev., **B13**, 316–374 (1976).
- [3] McCoy, B. M., Tracy, C. A., and Wu, T. T.: J. Math. Phys., **18**, 1058–1092 (1977).