5. On the Nilpotency Indices of the Radicals of Group Algebras of p-Solvable Groups

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Let K be an algebraically closed field with characteristic $p>0$, G a finite group of order $p^m g'$, $(p, g')=1$, KG a group algebra of G over K, $J(KG)$ the radical of KG and $t(G)$ the nilpotency index of $J(KG)$.

For a block B of KG denote by $t(B)$ the nilpotency index of the radical $J(B)$ of B. G. O. Michler [6] showed that if a defect group D of B is cyclic and normal in G, then B is a serial ring and $t(B)=|D|$. In this paper we shall prove that when D is cyclic, B is serial if and only if $t(B)=|D|$.

D. S. Passman [9], Y. Tsushima [11] and D. A. R. Wallace [12] showed that $m(p-1)+1\leqq t(G)\leqq p^m$ provided G is p-solvable. Recently K. Motose and Y. Ninomiya [8] proved that for a p-solvable group G of p-length 1, $t(G)=p^m$ if and only if a p-Sylow subgroup P of G is cyclic. We shall generalize this result as follows: For an arbitrary p-solvable group G, $t(G)=p^m$ if and only if P is cyclic. This is an affirmative answer to Ninomiya's conjecture announced in the Summer Algebra Symposium at Matsuyama in Japan (1974).

We call a module *uniserial* if it has a unique composition series of finite length. To being with we shall prove

Proposition 1. *Let B be a block of KG with a defect group D. If D is cyclic, then $t(B)\leqq|D|$.*

Proof. We can assume that $J(B)\neq 0$. Put that $B=\sum_{i=1}^{n}\sum_{j=1}^{f_i}\oplus KGe_{ij}$, where $\{e_{ij}\}$ are orthogonal primitive idempotents of KG such that $KGe_{i1}\cong KGe_{ij}$ for $j=1,\cdots,f_i$; $i=1,\cdots,n$ and $KGe_{i1}\not\cong KGe_{k1}$ if $i\neq k$, and $e_{i1}=e_i$ for $i=1,\cdots,n$. Let $C=(c_{ik})_{1\leqq i,k\leqq n}$ be the Cartan matrix for B and t_i the least positive integer such that $J(KG)^{t_i}e_i=0$ for $i=1,\cdots,n$. Then $t(B)\leqq\max\{t_k\,|\,1\leqq k\leqq n\}=t_i$ for some i and $t_i\leqq s_i$, where $s_i=\sum_{k=1}^{n}c_{ik}$. By [4, Satz 1], there is a pair of uniserial left KG-modules L_{i1}, L_{i2} such that $J(KG)e_i=L_{i1}+L_{i2}$, $L_{i1}\cap L_{i2}\cong KGe_i/J(KG)e_i$, L_{i1} and L_{i2} have no common composition factors except $KGe_i/J(KG)e_i$, and all composition factors of L_{i1} are nonisomorphic. Again, by [4, Satz 1], $s_i=r_{i1}+(c_{ii}-1)r_{i2}$, where r_{iv} is the number of nonisomorphic composition factors of L_{iv} for $v=1,2$, and $r_{i1}+r_{i2}\leqq n+1$. If we put that $c=\max\{c_{kk}-1\,|\,1\leqq k\leqq n\}$, by [1, Theorem 1], $|D|=cn+1$. Therefore $t(B)\leqq|D|$.

Corollary 2. *Let P be a p-Sylow subgroup of G. If P is cyclic, then $t(G) \leqq |P|$.*

An artinian ring $R \ni 1$ is called *serial* if Re and eR are uniserial modules for any primitive idempotent e of R. Then we have

Theorem 3. *Let B be a block of KG with a defect group D. If D is cyclic, then the followings are equivalent.*

(1) $t(B)=|D|$.

(2) *B is a serial ring.*

(3) *The Cartan matrix for B has form*

$$\begin{bmatrix} c+1 & c & \cdot & \cdot & \cdot & c \\ c & c+1 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & c \\ c & \cdot & \cdot & \cdot & c & c+1 \end{bmatrix}.$$

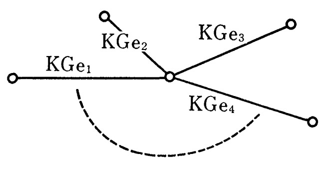
Proof. If $D=1$, B is simple artinian. So (1), (2) and (3) always hold. We can assume that $D \neq 1$. We use notations as in Proposition 1.

(2)$\Rightarrow$(3): This follows from [7, Lemma 1].

(3)$\Rightarrow$(1): By (3) and [1, Theorem 1], $s_i=cn+1=|D|$ for all i. From [5, Folgerung 4], B is serial. So $KGe_i \supsetneqq J(KG)e_i \supsetneqq \cdots \supsetneqq J(KG)^{|D|}e_i =0$ is a unique composition series of KGe_i for all i. Put that $E =\sum_{i=1}^{n} \sum_{j=1}^{f_i} e_{ij}$. Since $E \in Z(KG)$ and E is a unit element of B, $0 \neq J(KG)^{|D|-1}e_1 = EJ(KG)^{|D|-1}e_1E \subseteqq EJ(KG)^{|D|-1}E = (EJ(KG)E)^{|D|-1} =J(B)^{|D|-1}$. Hence from Proposition 1, $t(B)=|D|$.

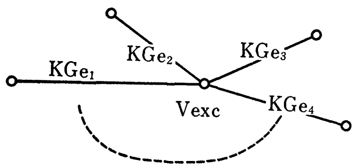
(1)$\Rightarrow$(2): By the proof of Propositon 1 and (1), $t(B)=t_i=s_i=cn+1 =|D|$ for some i. This shows that KGe_i is uniserial, and hence $J(KG)e_i =L_{i1}$ or $J(KG)e_i=L_{i2}$ by [4, Satz 1].

Case 1: Assume that $J(KG)e_i=L_{i1}$. Since $s_i=|D|$, by [4, Satz 1] and [1, Theorem 1], the number of nonisomorphic composition factors of KGe_i is equal to $|D|-1=cn$. From the definition of $n, c=1$. So $KGe_1/J(KG)e_1, \cdots, KGe_n/J(KG)e_n$ appear as composition factors of KGe_i. Hence, by rearranging the numbers $1, \cdots, n$, the Brauer tree of B has form



Since $c=1$, KGe_k is uniserial for all $k=1, \cdots, n$.

Case 2: Assume that $J(KG)e_i=L_{i2}$. As in Case 1, the number of nonisomorphic composition factors of KGe_i is equal to $(|D|-1)/(c_{ii}-1) =cn/(c_{ii}-1) \geqq n$. So as in Case 1, $c=c_{ii}-1$ and, by rearranging the numbers $1, \cdots, n$, the Brauer tree of B has form



where V_{exc} is the exceptional vertex. This shows that KGe_k is uniserial for all $k=1, \cdots, n$.

Similarly, e_kKG is uniserial for all $k=1, \cdots, n$. This completes the proof of Theorem 3.

Now, using Theorem 3 and group theory we have the following main theorem of this paper. This is a generalization of [8, Corollary 1 (2)].

Theorem 4. *Let G be a p-solvable group with a p-Sylow subgroup P. Then $t(G)=|P|$ if and only if P is cyclic.*

Proof. If P is cyclic, by [10, Theorem 3], KG is serial. By applying Theorem 3 for each block of KG, $t(G)=|P|$.

Conversely, assume that $t(G)=|P|=p^m$. We use induction on $|G|$. If $G=1$, it is trivial. Assume that $G\neq 1$ and it is proved for p-solvable groups of orders $1, \cdots, |G|-1$. From [8, Corollary 1], we may put that $m\geqq 3$. Since G is p-solvable, G has a proper normal subgroup H such that $(|G:H|, p)=1$ or $|G:H|=p$. If $(|G:H|, p)=1$, by [9, Lemma 1.2] and [9, Proposition 1.5], $t(G)=t(H)$. So from the hypothesis of induction, H has a cyclic p-Sylow subgroup. i.e. P is cyclic. Therefore we can assume that $|G:H|=p$. By [9, Proposition 1.3] and [9, Lemma 1.2], $J(KG)^p\subseteqq J(KH)KG=KG\cdot J(KH)$. So $J(KG)^{p\cdot t(H)}\subseteqq J(KH)^{t(H)}KG=0$. Hence $t(G)\leqq p\cdot t(H)$. From this and [9, Theorem 1.6], $t(H)=p^{m-1}$. Thus from the hypothesis of induction, H has a cyclic p-Sylow subgroup of order p^{m-1}. Now, suppose that P is not cyclic. By [8, Corollary 1], P is not abelian. Since P has a cyclic subgroup of index p, by [2, Chap. 5 Theorem 4.4], P is one of the following types,

(i) $p\geqq 3$ and $P\cong M_m(p)=\langle a, b\,|\,a^p=b^{p^{m-1}}=1, a^{-1}ba=b^{p^{m-2}+1}\rangle$,

(ii) $p=2$, $m=3$ and $P\cong D_3$ or Q_3,

(iii) $p=2$, $m\geqq 4$ and $P\cong M_m(2)$, D_m, Q_m or S_m,

where $M_m(2)$ is defined for $p=2$ in (i), D_m is a dihedral group, Q_m is a generalized quaternion group and S_m is a semi-dihedral group (cf. [2, Chap. 2, Chap. 5]).

Case 1: Assume that $p=2$. Since H has a cyclic 2-Sylow subgroup, by [2, Chap. 7 Theorem 6.1], H has a normal 2-complement L. L is characteristic in H and H is normal in G, and hence L is normal in G. This implies that L is a normal 2-complement of G. So from [8, Corollary 1], $t(G)\neq 2^m$ since P is not cyclic. This is a contradiction.

Case 2: Assume that $p\geqq 3$. We can put that $P=M_m(p)$. We use notations P_i and N_i for G as in [3, § 1]. Since $Z(p)=\langle b^p\rangle$, by [2, Chap. 6 Theorem 3.3], $b^p \in 0_{p',p}(G)=P_1$. If $a \in P_1$, by [8, Corollary 1] and the hypothesis that P is not cyclic, $\langle a, b^p\rangle$ is a p-Sylow subgroup of P_1. So from [8, Corollary 1] and [9, Theorem 1.6], $t(P_1)<p^{m-1}$. By [9, Proposition 1.5] and [9, Lemma 1.2], $t(G)=t(P_2)$ and $t(N_1)=t(P_1)$. By [9, Proposition 1.3] and [9, Lemma 1.2], $t(P_2)\leqq p\cdot t(N_1)$. Thus $p^m = t(G) < pp^{m-1}=p^m$. This is a contradiction. So we can assume that $a \notin P_1$. Thus we can put that $P_1/N_0=\langle bN_0\rangle$ or $\langle b^pN_0\rangle$. Denote by $\Phi(X)$ the Frattini subgroup of X for a finite group X. Put that $\Phi(P_1/N_0) =F/N_0$ and $S=\{x \in G \mid x^{-1}yxF=yF$ for all $y \in P_1\}$. If $P_1/N_0=\langle bN_0\rangle$, $F=\langle b^p\rangle N_0$. So from $a^{-1}b^ia=b^ib^{p^{m-2}i}$, $a^{-1}b^iaF=b^iF$ for all i. If $P_1/N_0 =\langle b^pN_0\rangle$, $F=\langle b^{p^2}\rangle N_0$. Since $Z(P)=\langle b^p\rangle$, $a^{-1}b^{pi}aF=b^{pi}F$ for all i. In any case $a \in S$. Hence, by [3, Lemma 1.2.5], $a \in P_1$. This is a contradiction. This finishes the proof of Theorem 4.

Added in proof. After submitted this paper the author received from Mr. K. Motose a preprint entitled "On radicals of principal blocks", in which it is quoted the validity of Proposition 1. Further it is reported in the same paper that our Theorem 4 has been proved by Y. Tsushima independently, however, it seems to us that Tsushima's proof is different from ours.

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