4. *Topologically Unequivalent Diffeomorphisms Whose Suspensions Are C^∞ Equivalent*

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Let $\psi_t(x, s) = (x, s+t)$ be the trivial flow on $M \times \boldsymbol{R}$. Let $M_f = M \times \boldsymbol{R}/(f(x), t) \sim (x, t+1)$ be the attaching torus of a diffeomorphism f on M. The flow φ_t on M_f induced by ψ_t is called a *suspension* of f.

If two diffeomorphisms f and f' on M and M', respectively, are C^r equivalent (C^r conjugate) the suspensions φ and φ' of f and f' are C^r equivalent; i.e. there is a C^r diffeomorphism from M_f to $M'_{f'}$ mapping any orbit of φ onto an orbit of φ' with preserving the orientations of orbits. But the converse is not true. (See [1] or [2].) In case that there is no surjection $\pi_1(M) \to \boldsymbol{Z}$ or $\pi_1(M') \to \boldsymbol{Z}$, the C^r equivalence of φ and φ' implies the C^r equivalence of f and f'. (See [1].)

M. M. Peixoto asked to the author whether there exist topologically unequivalent two diffeomorphisms on the same manifold whose suspensions are equivalent. Next theorem was motivated by this question.

Theorem. *Let N be a compact manifold with* $\dim N \geqq 0$ *and let $M = N \times S^1$, where S^1 is the 1-sphere. Then, there are infinitely many Morse-Smale C^∞ diffeomorphisms f_i $(i=1, 2, \cdots)$ on M satisfying the following properties.*

i) *The all suspensions of f_i $(i=1, 2, \cdots)$ are C^∞ equivalent.*

ii) *If $i \neq j$, f_i and f_j are not topologically equivalent.*

Lemma. *Let f be a diffeomorphism on $M = N \times S^1$ with at least one periodic point such that f is diffeotopic to the identity. (i.e. there is a smooth map $F: M \times I \to M$ such that $F(\ , 0) = id.$, $F(\ , 1) = f$, and that $F(\ , t)$ is a diffeomorphisms on M for any $t \in I$, where $I = [0, 1]$.) Then there are C^∞ diffeomorphisms f_i $(i=1, 2, \cdots)$ satisfying the following properties.*

i) $f_1 = f$.

ii) *The all suspensions of f_i $(i=1, 2, \cdots)$ are C^∞-equivalent.*

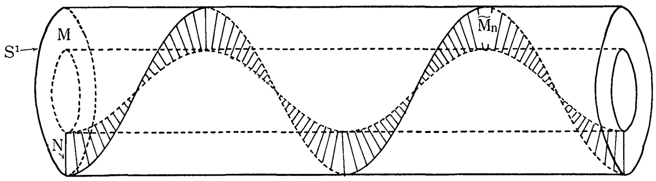
iii) *If $i \neq j$, f_i and f_j are not topologically equivalent.*

Proof. Since f is deffeotopic to the identity, M_f is diffeomorphic to $M \times S^1$. We may consider the suspension φ_t of f as a flow on $M \times S^1$ such that for any $s \in S^1 M \times s$ is a cross-section of φ_t. We define a submanifold M_n of $M \times S^1$ for $n = 2, 3, \cdots$ as follows.

$\tilde{M}_n = \{(x, e^{2\pi n t i}, t) \in N \times S^1 \times I \mid x \in N, t \in I\}$ is a codimension one submanifold of $M \times I$. By the attaching

$$M \times I/(x, 0) \sim (x, 1) = M \times S^1,$$

the submanifold $\tilde{M}_n$ of $M \times I$ becomes a submanifold M_n of $M \times S^1$. (See the figure.) For $n=1$, M_1 is defined as $M_1 = M \times 0 \subset M \times S^1$.



$M_1, M_2, \cdots$ are diffeomorphic to M. M_1 is a cross-section of φ_t and the Poincaré transformation is equal to f. Since M_n is a cross-section of φ_t for a sufficiently large n, we define f_n as the Poincaré transformation on M_n of φ_t. Let

$$P(g) = \inf \{\text{minimal period of } x \mid x \in \text{Per}(g)\}.$$

Since $P(f_n) = n \cdot P(f)$, f_i and f_j are not topologically equivalent for $i \neq j$. Therefore, for sufficiently large m $f_1, f_m, f_{m+1}, f_{m+2}, \cdots$ are the required diffeomorphisms.

Proof of Theorem. Let g be a time one diffeomorphism on M of the gradient vector field of a Morse function $\mu: M \to \boldsymbol{R}$. g has only finite periodic points and all of these are hyperbolic fixed points. Let $\psi: M \times \boldsymbol{R} \to M$ be the flow of grad μ. Then, $\psi \mid M \times I$ is a diffeotopy from the identity map to g. g can be approximated by a Morse-Smale diffeomorphism f. Since there is a diffeotopy from g to f, f is diffeotopic to the identity. The suspension φ of f is also Morse-Smale. Thus, for any cross-section of φ the Poincaré transformation is Morse-Smale. Hence, by the proof of Lemma all diffeomorphisms $f_1, f_2, \cdots$ obtained by Lemma are Morse-Smale. This proves Theorem.

$\mathcal{D}^\infty(M)$ and $\mathcal{X}^\infty(M)$ denote the spaces of all diffeomorphisms and vector fields on M with C^∞ topology.

Corollary (M. M. Peixoto). *For $M = N \times S^1$ there are infinitely many stable components in $\mathcal{D}^\infty(M)$ such that by suspension they go to the same stable component in $\mathcal{X}^\infty(M \times S^1)$.*

References

[1] G. Ikegami: On classification of dynamical systems with cross-section. Osaka J. Math., **6**, 419–433 (1969).

[2] ——: Flow equivalence of diffeomorphisms I, II. Osaka J. Math., **8**, 49–76 (1971); Cor., **9**, 335–336 (1972).