30. *On the Mixed Problem with d'Alembertian in a Quarter Space*

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(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1977)

Introduction. In this note we consider the mixed problem

$$(0.1)\quad \begin{cases} \Box u \equiv (D_t^2 - D_x^2 - \sum_{j=1}^{n-1} D_{y_j}^2)u = f(t,x,y) & \text{in } (0,\infty)\times \boldsymbol{R}_+^n, \\ Bu \equiv (D_x + b_0(t,y)D_t + \sum_{j=1}^{n-1} b_j(t,y)D_{y_j} + c(t,y))u|_{x=0} & \\ \quad = g(t,y) & \text{on } (0,\infty)\times \boldsymbol{R}^{n-1}, \\ D_t u|_{t=0} = u_1(x,y) & \text{on } \boldsymbol{R}_+^n, \\ u|_{t=0} = u_0(x,y) & \text{on } \boldsymbol{R}_+^n, \end{cases}$$

where $D_t = -i\partial/\partial t, D_x = -i\partial/\partial x, \cdots, c(t,y) \in \mathscr{B}^\infty(\bar{\boldsymbol{R}}_+^1 \times \boldsymbol{R}^{n-1})$[1] and $b_j(t,y)$ $(j=0,1,\cdots,n-1)$ are real-valued functions belonging to $\mathscr{B}^\infty(\bar{\boldsymbol{R}}_+^1 \times \boldsymbol{R}^{n-1})$. Let us say that (0.1) is C^∞ well-posed when there exists a unique solution $u(t,x,y)$ in $C^\infty(\bar{\boldsymbol{R}}_+^1 \times \bar{\boldsymbol{R}}_+^n)$ for any $(u_0, u_1, f, g) \in C^\infty(\bar{\boldsymbol{R}}_+^n) \times C^\infty(\bar{\boldsymbol{R}}_+^n) \times C^\infty(\bar{\boldsymbol{R}}_+^1 \times \bar{\boldsymbol{R}}_+^n) \times C^\infty(\bar{\boldsymbol{R}}_+^1 \times \boldsymbol{R}^{n-1})$ satisfying the compatibility condition of infinite order.

When $b_0, \cdots, b_{n-1}$ and c are all constant, by Sakamoto [4] we know a necessary and sufficient condition for C^∞ well-posedness. If $b_0<1$ (0.1) is C^∞ well-posed, and in the case $n \geqq 3$ it is so only if $b_0<1$. Agemi and Shirota in [1] studied (0.1) precisely when $n=2$, $c=0$ (b_j is constant). Tsuji in [6] treated the case that $b_0, \cdots, b_{n-1}$ and c are variable, and showed the existence of the solution in the Sobolev space. Furthermore, he stated that the Lopatinski condition must be satified at any point if (0.1) is C^∞ well-posed. Ikawa [2] investigated (0.1) in a general domain in the case $n=2$, $b_0=0$.

In our note we shall study C^∞ well-posedness and the propagation speed of (0.1). Consider the following equation in λ:

$$\sqrt{1-\lambda^2} = b_0(t,y) + |b'(t,y)|\lambda \qquad (b' = (b_1, \cdots, b_{n-1})).$$

Then, if $b_0(t,y)<1$ this equation has a positive root or no real root. In the former case we denote the positive root by $\lambda_0(t,y)$, and in the latter case set $\lambda_0(t,y)=1$.

Theorem 1. *If* $\sup_{(t,y)\in \boldsymbol{R}_+^1 \times \boldsymbol{R}^{n-1}} b_0(t,y) < 1$, *then* (0.1) *is* C^∞ *well-posed and has a finite propagation speed less than* $\sup_{(t,y)\in \boldsymbol{R}_+^1 \times \boldsymbol{R}^{n-1}} \lambda_0(t,y)^{-1}$.

For a constant $v>0$ we set $C_v(t_0, x_0, y_0) = \{(t,x,y) : (t-t_0)v + ((x-x_0)^2$

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1) $\mathscr{B}^\infty(M)$ denotes the set $\{h(z) \in C^\infty(M); |h|_m = \sum_{|\alpha|\leqq m} |D_z^\alpha h(z)| < \infty$ for $m=0,1,\cdots\}$.

$+|y-y_0|^2)^{1/2}<0\}$. Fix the point (t_0, x_0, y_0), and let us have constants v, $\delta(>0)$ such that $u(t,x,y)=0$ on $C_v(t_0,x_0,y_0)\cap\{0<t_0-t<\delta, x>0\}$ for any $u\in C^\infty(\bar{R}^1_+\times\bar{R}^n_+)$ satisfying $\Box u=0$ on $C_v\cap\{0<t_0-t<\delta, x>0\}$, $u|_{t=t_0-\delta}=D_t u|_{t=t_0-\delta}=0$ on $C_v\cap\{t=t_0-\delta, x>0\}$ and $Bu=0$ on $C_v\cap\{0<t_0-t<\delta, x=0\}$. Then we call the infimum of the v the propagation speed at (t_0, x_0, y_0).

Theorem 2. *Let* $\sup_{(t,y)\in R^1_+\times R^{n-1}} b_0(t,y)<1$. *The propagation speed of* (0.1) *at any* $(t_0, 0, y_0)$ *is not smaller than* $\lambda_0(t_0,y_0)^{-1}$.

The author wishes to express his sincere gratitude to Prof. M. Ikawa for his many useful suggestions.

§ 1. Reduction to the equation on the boundary. Let us prove Theorem 1. We assume that $b(z)=(b_0(z),\cdots,b_{n-1}(z))$ and $c(z)$ $(z=(t,y))$ are constant when $|z|$ is large. The general case is reduced to this case. Let $b(z)=b$ and $c(z)=\tilde{c}$ for $|z|\geqq z_0$ (z_0 is a large constant). Solving the Cauchy problem, we can assume in the problem (0.1) that $u_0=u_1=0, f=0$. Then the compatibility condition of infinite order implies that every $D_t^j g(+0,y)$ $(j=0,1,\cdots)$ equals zero. Denote by $C_+^\infty(R^n)$ the set of C^∞ functions in R^n whose support lies in $\{t_0\leqq t\}$ for some $t_0\in R$. We know that the Dirichlet problem

$$\begin{cases}\Box w(z,x)=0 & \text{in } R^n\times R^1_+,\\ w|_{x=0}=h(z) & \text{on } R^n\end{cases}$$

has a unique solution $w(z,x)$ in $C_+^\infty(R^n\times\bar{R}^1_+)$ for any $h(z)\in C_+^\infty(R^n)$ and has a finite propagation speed, which equals one. We set (for $h\in C_+^\infty(R^n)$)

$$Th=Bw.^{2)}$$

Theorem 1.1. *There exists a unique solution h of the equation $Th=g$ in $C_+^\infty(R^n)$ for any $g\in C_+^\infty(R^n)$, and it has a finite propagation speed less than* $\sup_{z\in R^n}\lambda_0(z)^{-1}$.

This theorem yields Theorem 1 in Introduction.

§ 2. Proof of Theorem 1.1. We denote by $H_{m,\gamma}(R^n)(\gamma\in R^n, m\in R)$ the functional space $\{u(z): e^{-\gamma z}u(z)\in H_m(R^n)\}$. Let us define the Laplace-Fourier transformation $F_\gamma(\gamma\in R^n)$ by

$$F_\gamma[u]=\hat{u}(\zeta)=\int e^{-i(\sigma-i\gamma)z}u(z)dz \qquad (\zeta=\sigma-i\gamma),\ u\in C_0^\infty(R^n),$$

and denote by $\bar{F}_\gamma$ the inverse transformation

$$\left(\text{i.e. } \bar{F}_\gamma[f](z)=(2\pi)^{-n}e^{\gamma z}\int e^{i\sigma z}f(\sigma-i\gamma)d\sigma\right).$$

The norm $\langle h\rangle_{m,\gamma}$ of $H_{m,\gamma}(R^n)$ is defined by

$$\langle h\rangle^2_{m,\gamma}=(2\pi)^{-n}\int|\sigma-i\gamma|^{2m}|\hat{h}(\sigma-i\gamma)|^2 d\sigma \qquad (\gamma\neq 0).$$

Proposition 2.1. *We have* $\tau^2-\sum_{j=1}^{n-1}\eta_j^2-\xi^2\neq 0$ *for* $(\tau,\eta,\xi)\in R^{n+1}$

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2) Let the coefficients of B be extended smoothly to $t<0$.

$-i\Gamma$ ($\Gamma=\{(\zeta,\xi)=(\tau,\eta,\xi)\in \boldsymbol{R}^{n+1};\ \tau>(|\eta|^2+\xi^2)^{1/2}\}$).

Corollary. *If* $(\tau,\eta)\in \boldsymbol{R}^n-i\dot{\Gamma}$ ($\dot{\Gamma}=\{\zeta=(\tau,\eta)\in \boldsymbol{R}^n;\ \tau>|\eta|\}$), *the equation* $\tau^2-\sum_{j=1}^{n-1}\eta_j^2-\xi^2=0$ *in* ξ *has a root* $\xi_+(\tau,\eta)$ *with a positive imaginary part and a one with negative imaginary part* (cf. § 3 of Sakamoto [4]).

Let us set (for $\gamma\in\dot{\Gamma}$ and $h\in C_0^\infty(\boldsymbol{R}^n)$)

$$R_\gamma h=\overline{F}_\gamma[(\xi_+(\sigma-i\gamma)+b\cdot(\sigma-i\gamma)+c)\hat{h}(\sigma-i\gamma)],$$

$$R^*_{-\gamma}h=\overline{F}_{-\gamma}[(\overline{\xi_+(\sigma-i\gamma)}+b\cdot(\sigma+i\gamma)+D_z\cdot b+\bar{c})\hat{h}(\sigma+i\gamma)].$$

Then we have $Th=R_\gamma h$ ($\gamma\in\dot{\Gamma}$) for $h\in C_0^\infty(\boldsymbol{R}^n)$ and have

$$(R_\gamma h,g)_{L^2}=(h,R^*_{-\gamma}g)_{L^2},\qquad h,g\in C_0^\infty(\boldsymbol{R}^n)\ (\gamma\in\dot{\Gamma}).$$

Lemma 2.1. *Let* $m\in\boldsymbol{R}$ *and* S *be any compact set of* $\dot{\Sigma}=\{\zeta=(\tau,\eta)\in\boldsymbol{R}^n;\ \tau>(\sup_{z\in\boldsymbol{R}^n}\lambda_0(z)^{-1})|\eta|\}$. *There is a constant* $\gamma_0(m,S)$ *such that if* $|\gamma|\geqq\gamma_0(m,S)$ *and* $\gamma\in K_S=\{\gamma=\mu\zeta:\zeta\in S,\mu>0\}$ *the following estimates hold:*

(i) $|\gamma|\langle h\rangle_{m,\gamma}\leqq C\langle R_\gamma h\rangle_{m,\gamma}$, $h\in C_0^\infty(\boldsymbol{R}^n)$,

(ii) $|\gamma|\langle h\rangle_{-m,-\gamma}\leqq C\langle R^*_{-\gamma}h\rangle_{-m,-\gamma}$, $h\in C_0^\infty(\boldsymbol{R}^n)$.

This lemma is proved by means of the following lemma.

Lemma 2.2. *Let* S *be a compact set in* $\dot{\Sigma}$. *Then there is a constant* δ (>0) *such that*

$$\operatorname{Im}\xi_+(\zeta)+b(z)\cdot\operatorname{Im}\zeta\geqq\delta|\operatorname{Im}\zeta|,\ \zeta\in\boldsymbol{R}^n-iK_S,\ z\in\boldsymbol{R}^n.$$

Proof. In view of the corollary of Proposition 2.1, we see that $(-\operatorname{Im}\zeta,-\operatorname{Im}\xi_+(\zeta))\notin\Gamma$ if $\zeta\in\boldsymbol{R}^n-i\dot{\Gamma}$. On the other hand, if $\gamma\in K_S$, $\xi<0$ and $(\gamma,\xi)\notin\Gamma$ there is a small constant $\delta(>0)$ such that $\xi\leqq-(b+\delta\omega)\cdot\gamma$ for any $\omega(\omega\in\boldsymbol{R}^n,|\omega|=1)$. Therefore we have

$$\operatorname{Im}\xi_+(\zeta)+(b-\delta\operatorname{Im}\zeta/|\operatorname{Im}\zeta|)\operatorname{Im}\zeta\geqq0,\ \zeta\in\boldsymbol{R}^n-iK_S,\ z\in\boldsymbol{R}^n.$$

Proof of Theorem 1.1. It suffices to show that for any $g\in H_m(\boldsymbol{R}^n)$ satisfying supp $[g]\subset\dot{\Sigma}'+z_1(z_1\in\boldsymbol{R}^n)$ there exists a solution $h(\in H_{m,\tilde{\gamma}}(\boldsymbol{R}^n)$, $\tilde{\gamma}\in\dot{\Sigma})$ of $R_{\tilde{\gamma}}h=g$ whose support lies in $\dot{\Sigma}'+z_1$. Here $\dot{\Sigma}'$ is the set $\{\gamma'\in\boldsymbol{R}^n;\ \gamma'\cdot\gamma>0$ for any $\gamma\in\dot{\Sigma}\}$. Lemma 2.1 yields a solution $h_{\tilde{\gamma}}\in H_{m,\tilde{\gamma}}(\boldsymbol{R}^n)$ satisfying $R^n_\gamma h_{\tilde{\gamma}}=g$ ($\tilde{\gamma}\in\dot{\Sigma}$ and $|\tilde{\gamma}|$ is sufficiently large). Set

$$\tilde{R}_\gamma f=\overline{F}_\gamma[(\xi_+(\zeta)+\tilde{b}\cdot\zeta+\tilde{c})\hat{f}(\zeta)]\qquad(\zeta=\sigma-i\gamma).$$

Then we can write

$$\tilde{R}_{\tilde{\gamma}}h_{\tilde{\gamma}}=(\tilde{b}-b(z))\cdot D_zh_{\tilde{\gamma}}+(\tilde{c}-c(z))h_{\tilde{\gamma}}+g.$$

The support of the right term lies in $\dot{\Sigma}'+\tilde{z}(\tilde{z}\in\boldsymbol{R}^n)$. Noting that $\tilde{b}$ and $\tilde{c}$ are constant, we see supp $[h_{\tilde{\gamma}}]\subset\dot{\Sigma}'+\tilde{z}$ by Paley-Wiener's theorem (cf. Sakamoto [4]). Therefore $h_{\tilde{\gamma}}\in\bigcap_{\gamma\in\dot{\Sigma}}H_{m,\gamma}(\boldsymbol{R}^n)$. Hence we have $|\gamma|\langle h_{\tilde{\gamma}}\rangle_{m,\gamma}\leqq C\langle g\rangle_{m,\gamma}$ for any large $|\gamma|(\gamma\in\dot{\Sigma})$, which implies supp $[h_{\tilde{\gamma}}]\subset\dot{\Sigma}'+z_1$.

§ 3. Sketch of proof of Theorem 2. Theorem 2 is proved in the same way as in the proof of Theorem 4.1 of [5]. The idea of the proof is suggested by Kajitani [3] and Appendix of Ikawa [2]. Assume that there are positive constants δ and v ($<\lambda_0(t_0,y_0)^{-1}$) such that $u(t,x,y)=0$ on $C_v\cap\{0<t_0-t<\delta,x>0\}$ for any $u\in C^\infty(\bar{\boldsymbol{R}}^1_+\times\bar{\boldsymbol{R}}^n_+)$ satisfying $\square u=0$ on $C_v\cap\{0<t_0-t<\delta,x>0\}$, $u|_{t=t_0-\delta}=D_tu|_{t=t_0-\delta}=0$ on $C_v\cap\{t=t_0-\delta,x>0\}$

and $Bu=0$ on $C_v\cap\{0<t_0-t<\delta, x=0\}$. In order to show that this is a contradiction, we have only to construct an asymptotic solution $u_N(t, x, y)=\sum_{n=0}^{N} e^{ik\Phi(t,x,y)}v_n(t, x, y)(ik)^{-n}(k>0)$ such that $\square u_N=e^{ik\Phi}\square v_N \times(ik)^{-N}$ near $\bar{C}_v\cap\{0\leqq t_0-t\leqq\delta, x\geqq 0\}$, $u_N|_{t=t_0-\delta}=D_t u_N|_{t=t_0-\delta}=0$ on $C_v\cap\{t=t_0-\delta, x>0\}$, $Bu_N=0$ on $C_v\cap\{0<t_0-t<\delta, x=0\}$ and $v_0(t_0, 0, y_0)\neq 0$. Therefore we have the eiconal equation with $B\Phi=0$ and the transport equation with $Bv_n=0$. From the latter we get the equation for $v_n|_{x=0}$. Let $(1, a)(\in \boldsymbol{R}^n_{(t,y)})$ be the direction of the characteristic curve of this equation at (t_0, y_0). Then, choosing the phase function Φ appropriately, we have $|a|=\lambda_0(t_0, y_0)^{-1}$. Thus the required asymptotic solution is obtained.

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