On countability of Teichmüller modular groups for analytically infinite Riemann surfaces defined by generalized Cantor sets

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Abstract: For any analytically finite Riemann surface, the Teichmüller modular group is countable, but it is not easy to find an analytically infinite Riemann surface for which the Teichmüller modular group is countable. In this paper, we determine whether the Teichmüller modular group is countable or uncountable for some analytically infinite Riemann surfaces defined by generalized Cantor sets.

Key words: Riemann surface of infinite type; Teichmüller modular group; Generalized Cantor set.

1. Introduction.

1.1. Terminology of Riemann surfaces. We call a Riemann surface X *hyperbolic* if X is represented by a quotient space $\mathbf{D}/\Gamma$ of the unit disk $\mathbf{D}$ by a torsion-free Fuchsian group Γ. In this paper, any Riemann surface is supposed to be hyperbolic. A Riemann surface X is of *analytically finite type* if X is obtained from a compact surface by removing at most finitely many points, and X is of *analytically infinite type* if X is not of analytically finite type. On the other hand, a Riemann surface X is of *topologically finite type* if the fundamental group $\pi_1(X) \cong \Gamma$ is finitely generated, and X is of *topologically infinite type* if X is not of topologically finite type. Also, a Fuchsian group Γ is of *the first kind* if the limit set of Γ coincides with the unit circle: $\Lambda(\Gamma) = \partial\mathbf{D}$, and Γ is of *the second kind* if $\Lambda(\Gamma) \subsetneq \partial\mathbf{D}$. Now, a Fuchsian group Γ acts properly discontinuously on $\overline{\mathbf{D}} \setminus \Lambda(\Gamma)$, so if Γ is of the second kind, then we obtain a bordered Riemann surface $(\overline{\mathbf{D}} \setminus \Lambda(\Gamma))/\Gamma$ containing X as its interior. We refer to $(\partial\mathbf{D} \setminus \Lambda(\Gamma))/\Gamma$ as the *boundary at infinity* of X and write it as $\partial_\infty X$.

1.2. Teichmüller space and its Teichmüller modular group. For a Riemann surface X, the *Teichmüller space* $T(X)$ is the set of Teichmüller equivalence classes of quasiconformal mappings f of X onto another Riemann surface, where two quasiconformal mappings f_1 and f_2 are Teichmüller equivalent if there exists a conformal mapping $h : f_1(X) \to f_2(X)$ such that $f_2^{-1} \circ h \circ f_1 : X \to X$ is homotopic to the identity. If $\partial_\infty X \neq \emptyset$, the homotopy is considered to be relative to $\partial_\infty X$ ($:= rel.\ \partial_\infty X$), that is, the homotopy fixes points of $\partial_\infty X$. We write the Teichmüller equivalence class of f as $[f]$. It is known that $T(X)$ has a complex Banach manifold structure, and if X is of analytically finite type, then $\dim T(X) < \infty$; otherwise $\dim T(X) = \infty$. On $T(X)$, a distance between two points $[f_1]$ and $[f_2]$ is defined by $d_T([f_1],[f_2]) = \inf_f \log K(f)$, where the infimum is taken over all quasiconformal mappings from $f_1(X)$ to $f_2(X)$ homotopic to $f_2 \circ f_1^{-1}$ ($rel.\ \partial_\infty X$ if $\partial_\infty X \neq \emptyset$), and $K(f)$ is the maximal dilatation of f. This is a complete distance on $T(X)$ and is called the *Teichmüller distance*.

For a Riemann surface X, the *quasiconformal mapping class group* $\mathrm{MCG}(X)$ is defined as the group of all homotopy classes $[g]$ of quasiconformal automorphisms g of X ($rel.\ \partial_\infty X$ if $\partial_\infty X \neq \emptyset$). For each $[g] \in \mathrm{MCG}(X)$, we define the transformation $[g]_*$ of $T(X)$ as $[f] \mapsto [f \circ g^{-1}]$. Then $\mathrm{MCG}(X)$ acts on $T(X)$ isometrically with respect to d_T. Now, let $\mathrm{Aut}(T(X))$ be the group of all isometric biholomorphic automorphisms of $T(X)$. We consider the homomorphism $\iota : \mathrm{MCG}(X) \to \mathrm{Aut}(T(X))$ defined by $[g] \mapsto [g]_*$ and define the *Teichmüller modular group* for X, which is denoted by $\mathrm{Mod}(X)$, as the image $\mathrm{Im}\ \iota \subset \mathrm{Aut}(T(X))$ of ι. Except for a few low-dimensional Teichmüller spaces, the homomorphism ι is injective (cf. [2], [9]) and surjective (cf. [6]). Therefore, in this paper, we identify the quasi-

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conformal mapping class group with the Teichmüller modular group.

In section 3, we think a bit about the *reduced Teichmüller modular group* $\mathrm{Mod}^\sharp(X)$ for a bordered Riemann surface X. This is the quotient group of $\mathrm{Mod}(X)$ by free homotopy equivalence, that is, the homotopy does not necessarily fix points of the boundary of X.

1.3. Some Riemann surfaces of topologically infinite type and Teichmüller modular groups for them. In 2003, Shiga [10] considered two distances on the Teichmüller space $T(X)$; the Teichmüller distance d_T and the length spectrum distance d_L. By the definition, the Teichmüller distance $d_T([f_1],[f_2])$ means how different the complex structures of two Riemann surfaces $f_1(X)$ and $f_2(X)$ are. On the other hand, though we do not describe the definition in this paper, the length spectrum distance $d_L([f_1],[f_2])$ means how different the hyperbolic structures of two Riemann surfaces $f_1(X)$ and $f_2(X)$ are. If X is an analytically finite Riemann surface, then the two distances d_T and d_L define the same topology on $T(X)$, but otherwise it is not always true. Shiga constructed a topologically infinite Riemann surface S such that the two distances define different topologies on $T(S)$. His Riemann surface is essentially the same as the Riemann surface S constructed as follows: let $\{a_n\}_{n=1}^\infty$ be a monotonic divergent sequence of positive numbers such that $a_{n+1} > na_n$, and let $\{P_n\}_{n=1}^\infty$ be a sequence of pairs of pants such that the hyperbolic lengths of three boundary geodesics of P_n are a_n, a_{n+1}, a_{n+1} ($n = 1, 2, ...$). Firstly, make 2 copies of P_1 and glue them together along the boundaries of length a_1, then we obtain a Riemann surface S_1 of type $(0, 4)$. Secondly, make 4 copies of P_2 and glue them to S_1 along the boundaries of length a_2, then we obtain a Riemann surface S_2 of type $(0, 8)$. Inductively, for each n, make 2^n copies of P_n and glue them to S_{n-1} along the boundaries of length a_n, then we obtain a Riemann surface S_n of type $(0, 2^{n+1})$. We define the Riemann surface S as the exhaustion of $\{S_n\}_{n=1}^\infty$. Then the convex core of S is $\bigcup_{n=1}^\infty S_n$. (He also showed that if a topologically infinite Riemann surface X satisfies some condition, the two distances define the same topology on $T(X)$ in the same paper [10]. And in 2018, we generalized his theorem, more precisely, we showed that if X is a Riemann surface with bounded geometry, then the two distances define the same topology on $T(X)$ [4].)

In 2004, Matsuzaki [8] considered Shiga's Riemann surface S, a reconstructed Riemann surface R from S and the Teichmüller modular group $\mathrm{Mod}(R)$ for R. Before mentioning it, we introduce a proposition for countability of the Teichmüller modular group.

Proposition 1.1 (Proposition 1 of [8]). *Suppose X is a hyperbolic Riemann surface. If* $\mathrm{Mod}(X)$ is countable, then $X = \mathbf{D}/\Gamma$ satisfies the following conditions.

(1) The number of simple closed geodesics on X whose lengths are smaller than M for arbitrary $M > 0$ is finite.
(2) The Fuchsian group Γ is of the first kind.

In §3 of [8], Matsuzaki showed that if a Riemann surface S is constructed by gluing above-mentioned pants $\{P_n\}_{n=1}^\infty$ in the *usual* way, then S is not geodesically complete, that is, there exists a geodesic connecting ∂P_1 and ∂P_n such that its length converges as $n \to \infty$. This means that the geodesic completion of S does not coincide with S, hence the Fuchsian group corresponding to S is of the second kind. (cf. Proposition 3.7 of [1].) In particular, $\mathrm{Mod}(S)$ is uncountable by Proposition 1.1 (2). However, if a Riemann surface R is constructed by gluing the aforementioned pants $\{P_n\}_{n=1}^\infty$ in a *special* way, then R is geodesically complete, so the geodesic completion of R coincides with R. Here, a special way is to give each boundary geodesic of each pair of pants some amount of twist when we glue pants together. Then, the corresponding Fuchsian group is of the first kind, and also he could show that $\mathrm{Mod}(R)$ is countable.

1.4. Generalized Cantor sets. Let $\{q_n\}_{n=1}^\infty$ be a sequence of numbers in $(0, 1)$. Put $I := [0, 1] \subset \mathbf{R}$. A generalized Cantor set $E(\omega)$ for $\omega = \{q_n\}_{n=1}^\infty$ is defined as follows: Firstly, remove an open interval with the length q_1 from I so that the remaining intervals $I_1^1, I_1^2 \subset I$ have the same length. Secondly, remove an open interval with the length $q_2|I_1^1|$ from each I_1^i ($i = 1, 2$) so that the remaining intervals $I_2^1, I_2^2, I_2^3, I_2^4 \subset I$ have the same length, where $|\cdot|$ means the length of the interval. Inductively, remove an open interval with the length $q_n|I_{n-1}^1|$ from each I_{n-1}^i ($i = 1, ..., 2^{n-1}$) so that the remaining intervals $I_n^1, ..., I_n^{2^n} \subset I$ have the same length. For each $n \in \mathbf{N}$, put $E_n = \bigcup_{i=1}^{2^n} I_n^i$. We define a generalized Cantor set $E(\omega)$ for ω as $\bigcap_{n=1}^\infty E_n$. In our previous paper [5], we considered the Riemann surface $X_{E(\omega)} := \hat{\mathbf{C}} \setminus E(\omega)$ (obtained from the Riemann sphere $\hat{\mathbf{C}}$ by removing $E(\omega)$) and the Teichmüller space $T(X_{E(\omega)})$ of $X_{E(\omega)}$, and proved a theorem about the Teichmüller distance d_T and the length spectrum distance d_L on $T(X_{E(\omega)})$. In this paper, we consider the Teichmüller modular group for $X_{E(\omega)}$.

1.5. Our results. At first, we give a sufficient condition for $\mathrm{Mod}(X_{E(\omega)})$ to be uncountable. It is obtained by Proposition 1.1 above and a lemma of our previous paper [5].

Theorem 1.2. *If there exists a subsequence $\{q_{n(k)}\}_{k=1}^{\infty}$ of $\omega = \{q_n\}_{n=1}^{\infty}$ such that $q_{n(k)} > c$ for some constant $c \in (0,1)$, then the Teichmüller modular group for the Riemann surface $X_{E(\omega)}$ is uncountable. In particular, if $\inf_n q_n \neq 0$, then* $\mathrm{Mod}(X_{E(\omega)})$ is uncountable.

Not only ω such that $\inf_n q_n \neq 0$ but also some ω such that $\inf_n q_n = 0$ satisfies the condition of Theorem 1.2. For example, let $\omega = \{q_n\}_{n=1}^{\infty}$ be a sequence defined by

$$q_n = \begin{cases} \frac{1}{2} & (n = 2m-1; m \in \mathbf{N}) \\ (\frac{1}{2})^n & (n = 2m; m \in \mathbf{N}). \end{cases}$$

Then $\inf_n q_n = 0$ and there exists a subsequence $\{q_{2m-1}\}_{m=1}^{\infty}$ of ω such that $q_{2m-1} > 1/3$.

Next, we give a sufficient condition for $\mathrm{Mod}(X_{E(\omega)})$ to be countable. In Theorem 1.1 of our previous paper [5], we considered two conditions (I),(II) for ω such that $\inf_n q_n = 0$, and showed that if ω satisfies either (I) or (II), then the two distances d_T and d_L define the different topologies on $T(X_{E(\omega)})$. Now, if ω satisfies (II), then it satisfies the condition of Theorem 1.2 above, too. On the other hand, if ω satisfies (I), then it does not do so. In this paper, our main theorem below says that if ω satisfies (I), then $\mathrm{Mod}(X_{E(\omega)})$ is countable:

Theorem 1.3. *If the sequence ω satisfying*

$$q_n \cdot \log(\log(1/q_{n+1})) \to \infty$$

as $n \to \infty$, then $\mathrm{Mod}(X_{E(\omega)})$ is countable.

The sequence ω satisfying Theorem 1.3 converges to 0 very rapidly. The following is an example of such sequences which is a little different from Example 1.2 of [5].

Example 1.4. Take a sequence $\omega = \{q_n\}_{n=1}^{\infty}$ so that $q_{n+1} = 1/\exp(n^{1/q_n})$. Then

$$q_n \cdot \log(\log(1/q_{n+1})) = q_n \cdot (1/q_n) \log n = \log n \to \infty$$

as $n \to \infty$.

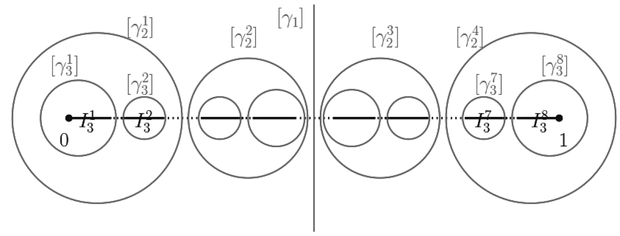

Fig. 1 Pairs of pants $\bigcup_{n=1}^{3}(\bigcup_{i=1}^{2^n} P_n^i)$

The advantage of considering $X_{E(\omega)}$ is the following

Proposition 1.5. *For any ω, the Fuchsian group Γ corresponding to $X_{E(\omega)}$ is of the first kind.*

By this property, we can construct the analytically infinite Riemann surface for which the Teichmüller modular group is countable without caring about twist of boundary geodesics of pairs of pants. In section 2, we prove Theorem 1.2 and Proposition 1.5. In section 3, we prove Theorem 1.3.

2. Proofs of Theorem 1.2 and Proposition 1.5. We decompose $X_{E(\omega)}$ into pairs of pants as we (§2 of [5]) or Shiga (§3 of [11]) did. Recall that for a sequence $\omega = \{q_n\}_{n=1}^{\infty}$, the generalized Cantor set $E(\omega)$ is $\bigcap_{n=1}^{\infty} E_n$, where E_n is the union of closed intervals $\{I_n^i\}_{i=1}^{2^n}$ in $I = [0,1]$ $(n = 1, 2, ...)$. Now, for each $n \in \mathbf{N}$ and each $i \in \{1, ..., 2^n\}$, let γ_n^i be a simple closed curve separating I_n^i from other intervals $I_n^{i'}$, $i' \in \{1, ..., 2^n\} \setminus \{i\}$. Then, in $E(\omega)$, let $[\gamma_n^i]$ be the simple closed geodesic which is freely homotopic to γ_n^i, where $[\gamma_1^1] = [\gamma_1^2]$, so we write $[\gamma_1]$ for the geodesic. Let P_1^i be a pair of pants with boundary geodesics $[\gamma_1]$, $[\gamma_2^{2i-1}]$ and $[\gamma_2^{2i}]$ $(i = 1, 2)$. And for each $n \in \mathbf{N}$ and $i \in \{1, ..., 2^n\}$, let P_n^i be a pair of pants with boundary geodesics $[\gamma_n^i]$, $[\gamma_{n+1}^{2i-1}]$ and $[\gamma_{n+1}^{2i}]$. (See Fig. 1.) Then we obtain a pants decomposition $\bigcup_{n=1}^{\infty}(\bigcup_{i=1}^{2^n} P_n^i)$. We call this a natural pants decomposition of $X_{E(\omega)}$.

To prove Theorem 1.2, we use a lemma in our previous paper [5]. Here, $\ell_X(\gamma)$ means the hyperbolic length of a curve γ on a hyperbolic Riemann surface X.

Lemma 2.1 (Lemma 2.1 (1) of [5]). *For any ω and any $n \geq 1$,*

$$\ell_{X_{E(\omega)}}([\gamma_n^1]) < \frac{\pi^2}{\tanh^{-1} q_n}$$

holds.

Remark 1. For each n, there are 2^n simple closed geodesics $\{[\gamma_n^i]\}_{i=1}^{2^n}$ as the n-th geodesic, and in the case where $2 \le i \le 2^{n-1}$, $\ell_{X_{E(\omega)}}([\gamma_n^i]) < \pi^2/\tanh^{-1} q_n$ does not hold, in general. (cf. Lemma 2.1 (2) of [5].) However, if ω is monotonic decreasing, then $\ell_{X_{E(\omega)}}([\gamma_n^i]) < \pi^2/\tanh^{-1} q_n$ holds for any $n \in \mathbf{N}$ and $i \in \{1,...,2^n\}$. (cf. Remark 3 of [5].)

From the above, Theorem 1.2 is immediately proved.

Proof of Theorem 1.2. By Lemma 2.1, if there exists a subsequence $\{q_{n(k)}\}_{k=1}^{\infty}$ of ω such that $q_{n(k)} > c$, then $\ell_{X_{E(\omega)}}([\gamma_{n(k)}^1]) < \pi^2/\tanh^{-1} c$ for $k = 1, 2, \ldots$ Hence, by Proposition 1.1 (1), $\mathrm{Mod}(X_{E(\omega)})$ is uncountable. □

Before proving Theorem 1.3, we check Proposition 1.5. The following proof is based on the idea of Prof. Shiga.

Proof of Proposition 1.5. Assume that the Fuchsian group Γ is of the second kind for some ω. Then, for any point $p \in \mathbf{D}$, there exists a geodesic ray $\hat{r}_p$ in $\mathbf{D}$ starting at p such that $\ell_{\mathbf{D}}(\hat{r}_p)$ is infinite, but $\ell_{X_{E(\omega)}}(\pi(\hat{r}_p) \cap \bigcup_{n=1}^{\infty}(\bigcup_{i=1}^{2^n} P_n^i))$ is finite, where $\pi : \mathbf{D} \to \mathbf{D}/\Gamma$ is the the universal covering and $\bigcup_{n=1}^{\infty}(\bigcup_{i=1}^{2^n} P_n^i)$ is a natural pants decomposition of $X_{E(\omega)}$. Put $r_p := \pi(\hat{r}_p)$. Let $\{P_n^*\}_{n=k_p}^{\infty}$ be a family of the pants containing r_p and $\{\gamma_n\}_{n=k_p}^{\infty}$ be a simple closed geodesic of each ∂P_n^* intersecting r_p. Here, $E(\omega)$ is totally disconnected, hence the diameter of γ_n in $\mathbf{C}$ converges to 0 as $n \to \infty$, and there exists a point $p_\infty \in E(\omega)$ such that γ_n converges to p_∞. Now take points $p_1, p_2 \in E(\omega)$ which are contained in $\hat{\mathbf{C}} \setminus \bigcup_{n=k_p}^{\infty} P_n^*$ and put $W := \hat{\mathbf{C}} \setminus \{p_\infty, p_1, p_2\}$. Since r_p goes to p_∞, we have $\ell_W(r_p) = \infty$. However, $X_{E(\omega)} \subset W$, so $\ell_{X_{E(\omega)}}(r_p) \ge \ell_W(r_p)$ and this is a contradiction. □

3. Proof of Theorem 1.3.

Let $\omega = \{q_n\}_n^{\infty}$ be a sequence of numbers in $(0,1)$, and let $[\gamma_n^i]$ and P_n^i $(n \in \mathbf{N},\ i \in \{1,...,2^n\})$ be a closed geodesic and a pair of pants of $X_{E(\omega)}$ taken in Section 2, respectively. For each $n \in \mathbf{N}$, we take the subsurface $X_n := \bigcup_{k=1}^{n}(\bigcup_{i=1}^{2^k} P_k^i)$ of $X_{E(\omega)}$. Firstly, we show the following lemma.

Lemma 3.1. *Let ω be the sequence satisfying the condition of Theorem 1.3. Then, for any K-quasiconformal automorphism $g : X_{E(\omega)} \to X_{E(\omega)}$, there exists $n_1 \in \mathbf{N}$ such that if $n \ge n_1$, then the image $g(X_n)$ of X_n is freely homotopic to X_n in $X_{E(\omega)}$, that is, each component of $\partial g(X_n)$ is homotopic to some component of ∂X_n.*

To prove this, we use Lemma 3.2 [5], Lemma 3.3 [5] and Lemma 3.4 (Wolpert's Lemma).

Lemma 3.2 (Lemma 2.3 of [5]). *Let ω be the sequence satisfying the condition of Theorem 1.3. Then, for any $i \in \{1,...,2^n\}$ and $j \in \{1,...,2^{n+1}\}$,*

$$\frac{\ell_{X_{E(\omega)}}([\gamma_{n+1}^j])}{\ell_{X_{E(\omega)}}([\gamma_n^i])} \to \infty$$

as $n \to \infty$.

Below, $s_n^i (\subset \mathbf{R})$ is the shortest geodesic segment connecting $[\gamma_{n+1}^{2i-1}]$ and $[\gamma_{n+1}^{2i}]$ in each pair of pants P_n^i with boundary geodesics $\{[\gamma_n^i], [\gamma_{n+1}^{2i-1}], [\gamma_{n+1}^{2i}]\}$, and $d(\cdot,\cdot)$ is the hyperbolic distance on $X_{E(\omega)}$.

Lemma 3.3 (Lemma 2.4 of [5]). *Let ω be the sequence satisfying the condition of Theorem 1.3. Then, for any $i \in \{1,...,2^n\}$,*

$$\frac{d([\gamma_n^i], s_n^i)}{\ell_{X_{E(\omega)}}([\gamma_n^i])} \to \infty$$

as $n \to \infty$.

Lemma 3.4 [12]. *Let $f : X \to X'$ be a K-quasiconformal homeomorphism from a hyperbolic Riemann surface X onto another hyperbolic Riemann surface X'. And let γ be a simple closed geodesic on X and $[f(\gamma)]$ be the geodesic of the free homotopy class of $f(\gamma)$. Then*

$$\frac{1}{K} \le \frac{\ell_{X'}([f(\gamma)])}{\ell_X([\gamma])} \le K.$$

Proof of Lemma 3.1. Note that for $K \ge 1$, there exists $n_1 \in \mathbf{N}$ such that if $n \ge n_1$, then

$$\frac{d([\gamma_n^i], s_n^i)}{\ell_{X_{E(\omega)}}([\gamma_n^i])} > K$$

for any $i \in \{1,...,2^n\}$ by Lemma 3.3. Now, assume that for any $N \in \mathbf{N}$, $g(X_n)$ is not freely homotopic to X_n in $X_{E(\omega)}$ for some $n \ge N$. Then, for some $n \ge n_1$, $g(X_n)$ is not freely homotopic to X_n, so there exists a component γ_n of ∂X_n such that $[g(\gamma_n)]$ crosses ∂X_n, where $[g(\gamma_n)]$ is the closed geodesic freely homotopic to $g(\gamma_n)$. Then, the length of $[g(\gamma_n)]$ is larger than $d([\gamma_{n+m}^j], s_{n+m}^j)$ for some $m \ge 0$ and $j \in \{1,...,2^{n+m}\}$ since $[g(\gamma_n)]$ crosses $[\gamma_{n+m}^j]$ and s_{n+m}^j. Therefore, $\ell_{X_{E(\omega)}}([g(\gamma_n)]) > d([\gamma_{n+m}^j], s_{n+m}^j) > K\ell_{X_{E(\omega)}}([\gamma_{n+m}^j]) \ge K\ell_{X_{E(\omega)}}([\gamma_n])$ holds by Lemmas 3.3 and 3.2. This contradicts Lemma 3.4. □

Next, we consider a half Dehn twist, a Dehn twist and multiple twists about each component of ∂X_n for a sufficiently large number n. We do not mention the definition of a half Dehn twist, but roughly speaking, a half Dehn twist about $[\gamma_n^i]$ is a (homotopy class of) quasiconformal automorphism of $X_{E(\omega)}$ interchanging two points of the boundary of $X_{E(\omega)}$, i.e., two points of $E(\omega)$. (To learn about a half (Dehn) twist, see [3], for example.)

Lemma 3.5. *Let ω be the sequence satisfying the condition of Theorem 1.3. For for any K-quasiconformal automorphism $g : X_{E(\omega)} \to X_{E(\omega)}$, there exists $n_2 \in \mathbf{N}$ such that if $n \geq n_2$, on any component of ∂X_n, g causes neither a half Dehn twist, a Dehn twist nor multiple twists.*

We use a lemma of our previous paper [5] and a theorem of Matsuzaki [7]. In the following, η is the collar function: $\eta(x) = \sinh^{-1}(1/\sinh(x/2))$.

Lemma 3.6 (Lemma 2.2 of [5]). *Let $\omega = \{q_n\}_{n=1}^{\infty}$ be an arbitrary sequence of numbers in $(0, 1)$. For any $n \in \mathbf{N}$ and $i \in \{1, ..., 2^n\}$,*

$$\ell_{X_{E(\omega)}}([\gamma_n^i]) > 2\eta\left(\frac{\pi^2}{\log((1+q_n)/(2q_n))}\right)$$

holds.

Theorem 3.7 (Part of Theorem 1 of [7]). *Let γ be a simple closed geodesic on a hyperbolic Riemann surface X and f be n-times Dehn twist about γ. Then the maximal dilatation of an extremal quasiconformal automorphism of f satisfies*

$$K(f) \geq \sqrt{\{(2|n|-1)\ell_X(\gamma)/\pi\}^2 + 1}.$$

Proof of Lemma 3.5. Since ω satisfies the condition of Theorem 1.3, $q_n \to 0$ as $n \to \infty$. Therefore $\ell_{X_{E(\omega)}}([\gamma_n^i]) \to \infty$ for any i by Lemma 3.6. Hence, for $K \geq 1$, there exists $n_2 \in \mathbf{N}$ such that $\ell_{X_{E(\omega)}}([\gamma_n^i]) > \pi K^2$ if $n \geq n_2$. Assume that g cause a half Dehn twist f_n on some component γ_n of ∂X_n for some $n \geq n_2$. Then $f_n^2 := f_n \circ f_n$ is a Dehn twist about γ_n, so the maximal dilatation $K(f_n^2)$ of f_n^2 is larger than $\sqrt{K^4+1} > K^2$ by Theorem 3.7. Since $K(f_n^2) \leq K(f_n)^2$, we have $K < K(f_n)$, and this is a contradiction. This also implies that g causes neither a Dehn twist nor multiple twists. □

Finally we prove the main theorem.

Proof of Theorem 1.3. For a K-quasiconformal automorphism $g : X_{E(\omega)} \to X_{E(\omega)}$, put $N := \max\{n_1, n_2\}$, where n_1, n_2 are numbers of Lemma 3.1 and Lemma 3.5, respectively. Then, for any $n \geq N$, $g(X_n \setminus X_{n-1})$ is homotopic to $X_n \setminus X_{n-1} = \bigsqcup_{i=1}^{2^n} P_n^i$ in $X_{E(\omega)}$, and on any component of ∂X_n, g causes neither a half Dehn twist, a Dehn twist nor multiple twists.

Now, for an arbitrary $K \in \mathbf{N}$, let $\mathrm{Mod}(X_{E(\omega)})_K$ be a subset of the Teichmüller modular group $\mathrm{Mod}(X_{E(\omega)})$ such that each element has K-quasiconformal automorphism g as a representative. From the above argument, $\mathrm{Mod}(X_{E(\omega)})_K$ is embedded into the reduced Teichmüller modular group $\mathrm{Mod}^\sharp(X_N)$ for the bordered Riemann surface X_N. Indeed, g is determined by $g|_{X_N}$, that is, for quasiconformal automorphisms g_1, g_2 of $X_{E(\omega)}$, if $g_1|_{X_N} = g_2|_{X_N}$, then $[g_1] = [g_2]$.

Since X_N is topologically finite, $\mathrm{Mod}^\sharp(X_N)$ is finitely generated, thus countable. Hence $\mathrm{Mod}(X_{E(\omega)})_K$ is countable for any $K \in \mathbf{N}$, and $\mathrm{Mod}(X_{E(\omega)})$ is countable, too. □

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