A note on transcendental analytic functions with rational coefficients mapping Q into itself

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Abstract: In this note, the main focus is on a question about transcendental entire functions mapping $\mathbf{Q}$ into $\mathbf{Q}$ (which is related to a Mahler's problem). In particular, we prove that, for any $t > 0$, there is no a transcendental entire function $f \in \mathbf{Q}[[z]]$ such that $f(\mathbf{Q}) \subseteq \mathbf{Q}$ and whose denominator of $f(p/q)$ is $O(q^t)$, for all rational numbers p/q, with q sufficiently large.

Key words: Transcendental functions; rational functions; Mahler's question; Liouville numbers; Maillet's Property.

1. Introduction. Transcendental number theory began in 1844, when Liouville [2] proved the existence of transcendental numbers. In fact, he was able to explicit an infinite class of such numbers. These numbers are the well-known *Liouville numbers*: a real number ξ is called a Liouville number, if there exists a sequence of distinct rational numbers $(p_k/q_k)_k$, with $q_k > 1$, such that

$$0 < \left|\xi - \frac{p_k}{q_k}\right| < \frac{1}{q_k^{\omega_k}},$$

where ω_k tends to infinite as $k \to \infty$. The set of all Liouville numbers is denoted by $\mathbf{L}$.

In his pioneering book, Maillet [4], in 1906, proved that $f(\mathbf{L}) \subseteq \mathbf{L}$, for any non-constant rational function $f \in \mathbf{Q}(z)$. In light of this fact, in 1984, Mahler [3] raised the following question:

Question 1. Are there transcendental entire functions $f(z)$ such that if ξ is any Liouville number, then so is $f(\xi)$?

In 2015, Marques and Moreira [5] showed the existence of uncountably many transcendental entire functions f such that $f(\mathbf{Q}) \subseteq \mathbf{Q}$ and for which $\mathrm{den}(f(p/q)) < q^{8q^2}$, for all rational number p/q, with $q > 1$ (here, and in what follows, $\mathrm{den}(z)$ denotes the denominator of the irreducible rational number z). It follows from their argument that Question 1 has a positive answer if the following question also has:

Question 2. Are there transcendental entire functions $f(z)$ such that $f(\mathbf{Q}) \subseteq \mathbf{Q}$ and

$$\mathrm{den}(f(p/q)) = O(q^t),$$

for all rational numbers p/q, with q sufficiently large, where $t \geq 0$ is a given positive integer?

There are some progress in this question. For instance, in 2016, Marques, Ramirez and Silva [6] showed that there is 'no' transcendental entire function $f(z) \in \mathbf{Q}[[z]]$ such that $f(\mathbf{Q}) \subseteq \mathbf{Q}$ and $\mathrm{den}(f(p/q)) = o(q)$ (this is the case $t \in [0, 1)$ in Question 2). By using Whittaker's theory [7] of polynomial expansions of analytic functions, in 2020, Lelis and Marques [1] proved that the answer is also 'no' for the case in which $f(z) \in \mathbf{C}[[z]]$ and $t = 1$.

The goal of this note is to prove, in particular, the non-existence of functions as in Question 2 with rational coefficients. More precisely,

Theorem 1.1. *Let t be a positive integer and $\Omega \subseteq \mathbf{C}$ be a neighborhood of origin. Then, there is no a transcendental function $f(z) = \sum_{k\geq 0} a_k z^k \in \mathbf{C}[[z]]$, analytic in Ω, such that $a_k \in \mathbf{Q}$, for all $k \in [0, t]$, $f(\mathbf{Q} \cap \Omega) \subseteq \mathbf{Q}$ and*

$$\mathrm{den}(f(p/q)) = O(q^{t/2}),$$

for all rational numbers $p/q \in \Omega$, with q sufficiently large.

As an immediate consequence, we infer that

Corollary 1. *Let t be a positive integer. Then, there is no a transcendental entire function $f(z) \in \mathbf{Q}[[z]]$ such that $f(\mathbf{Q}) \subseteq \mathbf{Q}$ and*

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$$\mathrm{den}(f(p/q)) = O(q^t),$$

for all rational numbers p/q, with q sufficiently large.

2. The Proof of Theorem 1.1. In order to simplify the argument, and causing no loss of generality, we shall prove the theorem for $2t$ (instead of t). Aiming for a contradiction, let $t \geq 1$ be an integer and suppose that $f(z) \in \mathbf{C}[[z]]$ is a transcendental function which is analytic in the neighborhood $\Omega \subseteq \mathbf{C}$ of the origin given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

such that $a_k \in \mathbf{Q}$, for all $k \in [0, 2t]$, $f(\mathbf{Q} \cap \Omega) \subseteq \mathbf{Q}$ and $\mathrm{den}(f(p/q)) = O(q^t)$, for all $p/q \in \Omega \cap \mathbf{Q}$ with q sufficiently large. Note that, without loss of generality, we may assume that $f(0) = 0$, that is, $a_0 = 0$.

Let $F(z) \in \mathbf{Q}[z]$ be the polynomial given by

$$F(z) = \sum_{n=1}^{2t} a_n z^n.$$

Then, there exist polynomials $P(z)$ and $Q(z)$ in $\mathbf{Q}[z]$ of degrees at most t such that

$$R(z) := \frac{P(z)}{Q(z)} = F(z) + z^{2t+1}K(z),$$

where $K(z) \in \mathbf{C}[[z]]$ is analytic in a neighborhood of origin. Indeed, suppose that

$$Q(z) = q_0 + q_1 z + \cdots + q_t z^t,$$

and consider the product $Q(z)F(z)$ given by

$$Q(z)F(z) = b_1 z + \cdots + b_{3t} z^{3t}.$$

Therefore, we need to determine $q_0, q_1, \ldots, q_t$ in $\mathbf{Q}$, not all zero, such that $b_j = 0$, for all $j \in [t+1, 2t]$. In other words, we want a non-trivial rational solution for the $t \times (t+1)$ homogeneous linear system

$$\begin{cases} a_{2t}q_0 + a_{2t-1}q_1 + \cdots + a_t q_t = 0 \\ a_{2t-1}q_0 + a_{2t-2}q_1 + \cdots + a_{t-1}q_t = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{t+1}q_0 + a_t q_1 + \cdots + a_1 q_t = 0 \end{cases}$$

in the variables $q_0, q_1, \ldots, q_t$. Since the number of variables is larger than the number of equations, then a basic Linear Algebra result ensures the existence of a solution $(q_0, \ldots, q_t) \in \mathbf{Q}^{t+1} \backslash \{(0, \ldots, 0)\}$. Now, let $j \geq 0$ be the smallest non-negative integer such that $q_j \neq 0$. Thus, we may suppose that $q_j = 1$ (because $(0, \ldots, 0, 1, q_{j+1}/q_j, \ldots, q_t/q_j)$ is also a solution of the previous linear system). So, we define

$$P(z) := b_{j+1}z^{j+1} + b_{j+2}z^{j+2} + \cdots + b_t z^t.$$

Therefore

$$P(z) = Q(z)F(z) + z^{2t+1}S(z),$$

with $S(z) \in \mathbf{Q}[z]$ and so

$$R(z) = \frac{P(z)}{Q(z)} = F(z) + z^{2t+1-j}K(z), \tag{1}$$

where $K(z) = S(z)/(1 + q_{j+1}z + \cdots + q_t z^{t-j})$ is analytic in a neighborhood $\Omega' \subseteq \mathbf{C}$ of origin.

On the other hand, we have that

$$f(z) = F(z) + z^{2t+1}\tilde{K}(z), \tag{2}$$

where $\tilde{K}(z)$ is analytic in Ω. Let M' be the smallest positive integer such that $1/M' \in \Omega \cap \Omega'$. Thus, combining (1) together with (2) and for all positive integers $M > M'$, we get

$$\left| f\left(\frac{1}{M}\right) - R\left(\frac{1}{M}\right) \right| = \frac{1}{M^{2t+1-j}}\Theta(M), \tag{3}$$

where

$$\Theta(M) = \left| \frac{1}{M^j}\tilde{K}\left(\frac{1}{M}\right) - K\left(\frac{1}{M}\right) \right|.$$

Now, by hypothesis, $f(1/M) = A_M/B_M \in \mathbf{Q}$, with $\gcd(A_M, B_M) = 1$ and $B_M \leq C_1 M^t$ (for some positive constant C_1). Moreover, we have that

$$R\left(\frac{1}{M}\right) = \frac{b_{j+1}M^{t-j-1} + b_{j+2}M^{t-j-2} \cdots + b_t}{M^{t-j} + q_{j+1}M^{t-j-1} + \cdots + q_t}$$

is a rational number. Note that, if we suppose that $f(1/M) \neq R(1/M)$ for infinitely many integers $M > M'$, then

$$\left| f\left(\frac{1}{M}\right) - R\left(\frac{1}{M}\right) \right| \geq \frac{1}{C_2 M^{2t-j}}, \tag{4}$$

for all integer M sufficiently large (say $M > M_0 > M'$), where $C_2 = 2C_1$. By (3) and (4), we obtain that

$$\begin{aligned} \frac{1}{M^{2t-j}} &\ll \left| f\left(\frac{1}{M}\right) - R\left(\frac{1}{M}\right) \right| \\ &= \frac{1}{M^{2t+1-j}} \left| \frac{1}{M^j}\tilde{K}\left(\frac{1}{M}\right) - K\left(\frac{1}{M}\right) \right| \\ &\ll \frac{1}{M^{2t+1-j}} \end{aligned}$$

yielding to the absurdity that $M^{2t+1-j} \ll M^{2t-j}$, for infinitely many positive integers M for which $f(1/M) \neq R(1/M)$.

Therefore, $f(1/M) = R(1/M)$ for all large enough integer M, say $M \geq M_1$. Thus, $f(z)$ and $R(z)$ coincide in the set $\{1/M : M > \max\{M_0, M_1\}\} \subseteq \Omega \cap \Omega'$ which has a limit point. By the Identity Theorem for Analytic Functions, we infer that $f(z) = R(z)$, for all $z \in \Omega \cap \Omega'$ with contradicts the transcendence of $f(z)$. The proof is then complete. □

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