Simple proof of the global inverse function theorem via the Hopf–Rinow theorem

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Abstract: We explain that Hadamard's global inverse function theorem very simply follows from the Hopf–Rinow theorem of Riemannian geometry.

Key words: Global inverse function theorem; Hopf–Rinow theorem.

1. Global inverse function theorem. The purpose of this note is to explain a very simple proof of Hadamard's global inverse function theorem. For a smooth map $f = (f_1, \dots, f_n)\colon \mathbf{R}^n \to \mathbf{R}^n$ and $x \in \mathbf{R}^n$ we denote by $Df(x) = (\partial f_i/\partial x_j)_{i,j}$ the Jacobian matrix of f at x. For an $n \times n$ matrix A we denote its operator norm by $\|A\|$.

The following statement is known as the global inverse function theorem:

Theorem 1.1 (The Global inverse function theorem)**.** *Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map such that the Jacobian matrix $Df(x)$ is invertible at every $x \in \mathbf{R}^n$ and that $\|(Df(x))^{-1}\|$ is bounded over $x \in \mathbf{R}^n$. Then f is a diffeomorphism of $\mathbf{R}^n$ onto itself.*

This theorem is usually attributed to the paper of Hadamard [Had04] in the early 20th century. Several authors [Sch69, Gor72, Pla74, Mil84, Rab97, RS15] had provided its proofs and generalizations from various viewpoints. The global inverse function theorem was used for the study of oscillatory integral in [AF78, p. 302]. We can prove the Fundamental Theorem of Algebra by using the global inverse function theorem; see the book of Krantz–Parks [KP02, §6.2, Theorem 6.2.5].

The point of Theorem 1.1 is the assumption that $\|(Df(x))^{-1}\|$ is bounded over $\mathbf{R}^n$. If we remove it, then the statement is false in general. For example, consider the (complex) exponential function $\mathbf{C} \ni z \mapsto e^z \in \mathbf{C}$. Its Jacobian matrix (as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$) is

$$e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix}, \qquad (z = x + y\sqrt{-1}) \tag{1.1}$$

which is invertible everywhere. However e^z is neither injective nor surjective. The inverse of the matrix (1.1) is given by

$$e^{-x} \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}.$$

Its operator norm is e^{-x}, which is unbounded over $\mathbf{C}$.

We will show that Theorem 1.1 is a very easy consequence of the Hopf–Rinow theorem. The Hopf–Rinow theorem is a standard result of Riemannian geometry and (a part of) its statement is as follows (see e.g. [KN63, Chapter IV Theorems 4.1 and 4.2]):

Theorem 1.2 (The Hopf–Rinow theorem)**.** *Let M be a connected Riemannain manifold, and suppose that it is complete as a metric space (i.e., every Cauchy sequence converges).*
Then we have:

(1) *For any two points in M there exists a length-minimizing geodesic between them.*
(2) *For every point $p \in M$ the exponential map $\exp_p$ is defined all over the tangent space T_pM.*

Proof of Theorem 1.1. It is enough to show that f is a bijection from $\mathbf{R}^n$ onto $\mathbf{R}^n$. Let h be the Euclidean metric on $\mathbf{R}^n$. From the assumption, there is $c > 0$ such that $h(df(v), df(v)) \geq c \cdot h(v, v)$ for all tangent vectors $v \in T\mathbf{R}^n$. We define a Riemannian metric g on $\mathbf{R}^n$ by $g(u, v) = h(df_x(u), df_x(v))$ for $u, v \in T_x\mathbf{R}^n$. The map $f\colon (\mathbf{R}^n, g) \to (\mathbf{R}^n, h)$ is a local isometry (i.e. the pull-back of h by f is equal to g).

Let d be the distance function induced by g. Then we have $d(x, y) \geq \sqrt{c}\,|x - y|$, where $|x - y|$ is the standard Euclidean distance. In particular

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$(\mathbf{R}^n, d)$ is complete as a metric space. We apply the Hopf–Rinow theorem to $(\mathbf{R}^n, g)$.

- We prove that f is injective. Suppose x and y are two distinct points of $\mathbf{R}^n$. By the Hopf–Rinow theorem there exists a geodesic γ between them with respect to the metric g. Since f is a local isometry, $f \circ \gamma$ is a geodesic in $(\mathbf{R}^n, h)$. Hence it is the line segment between $f(x)$ and $f(y)$. The image of $f \circ \gamma$ is not a single point because $df_x: T_x\mathbf{R}^n \to T_{f(x)}\mathbf{R}^n$ is invertible. Therefore $f(x) \neq f(y)$.
- Next we prove that f is surjective. We assume $f(0) = 0$ for simplicity. Let $x \in \mathbf{R}^n$ be an arbitrary point. We define a line $\ell: \mathbf{R} \to \mathbf{R}^n$ by $\ell(t) = tx$. Let $v := \ell'(0)$ be the tangent vector of ℓ at the origin. Take $u \in T_0\mathbf{R}^n$ with $df_0(u) = v$, and consider the geodesic $\gamma(t) := \exp_0(tu)$ (with respect to the metric g). This is defined for all $t \in \mathbf{R}$ by the Hopf–Rinow theorem. Then $f \circ \gamma$ is a line of constant speed and $(f \circ \gamma)'(0) = v = \ell'(0)$. Thus $f \circ \gamma(t) = \ell(t)$. In particular we have $f(\gamma(1)) = x$. □

2. Discussions and remarks. (1) The above proof of the surjectivity of f is close to the standard proof of the following theorem of Riemannian geometry [KN63, Chapter IV Theorem 4.6]: *Let M and N be connected Riemannian manifolds of the same dimension, and suppose M is complete. Then any local isometry from M to N is a covering map.* (See also the book of Hermann [Her77, Chapter 21].) Therefore it is fair to say that Riemannian geometry provides a natural (and, in a sense, broader) framework for the global inverse function theorem. More general global inverse function theorems were developed by Gutiérrez–Biasi–Santos [GBS09] in the context of Riemannian geometry. Rabier [Rab97] developed a global inverse function theorem for Finsler manifolds.

(2) As far as the authors know, most known proofs of the global inverse function theorem use some topological argument (e.g. the covering space theory in [Pla74] or homotopy arguments in [Sch69, Mil84]). However the above proof does not use any topological argument. Moreover, the above proof of the surjectivity of f is constructive, at least in principle. If the map f is given by an explicit formula, then the geodesic $\gamma(t)$ is a solution of an explicit ordinary differential equation, and we can numerically solve it. In particular we can approximately calculate the point $\gamma(1)$ satisfying $f(\gamma(1)) = x$.

(3) The metric g was defined by $g(u, v) = h(df(u), df(v))$ in the above proof. The geodesic equation involves first order differentials of the metric g and hence second order differentials of the map f. Therefore we need to assume that f is at least C^2 maps. It is known that the global inverse function theorem holds for C^1 maps f as well [Sch69, p. 16]; our proof does not provide an optimal regularity. We think that the striking simplicity of the proof compensates for this drawback.

(4) The proof of the Hopf–Rinow theorem is not very easy. One can argue that our proof just translates one difficulty into another. (In some sense, our discovery is that the Hopf–Rinow theorem contains all the ingredients needed for the proof of the global inverse function theorem.) However the Hopf–Rinow theorem is certainly much more well-known than the global inverse function theorem. We think that it is nice to see that the Hopf–Rinow theorem has such an unexpected application.

(5) Plastock [Pla74, §3] studied generalizations of the global inverse function theorem motivated by the Hopf–Rinow theorem. Hence our paper is conceptually similar to [Pla74]. However our viewpoint is different from [Pla74]. The point of our paper is that the global inverse function theorem is a corollary of the Hopf–Rinow theorem. On the other hand, Plastock [Pla74, §3] did not use the Hopf–Rinow theorem itself. He developed a method motivated by the proof of the Hopf–Rinow theorem.

(6) Probably the most useful "application" of this paper is to use it for education. It may be nice to explain the global inverse function theorem as a corollary of the Hopf–Rinow theorem in an introductory course on Riemannian geometry. Indeed the content of this paper grew out of an undergraduate course conducted by Tsukamoto in Kyoto university. He posed a problem concerning the global inverse function theorem in an exercise course on geometry. Then Ohkita found a proof using the Hopf–Rinow theorem. His idea looked so beautiful that Tsukamoto thought that it should be

published. Therefore all the ideas of this paper are due to Ohkita. Tsukamoto just elaborated technical details and presentations.

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