On Kakeya's maximal function II: high dimensional cases

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(Communicated by Masaki KASHIWARA, M.J.A., Jan. 12, 2024)

Abstract: In this paper an estimate of the Kakeya maximal operator on the high dimensional Euclidean spaces is given, that is, the Kakeya maximal operator is of restricted weak type $(d/(d-1), d/(d-1))$ for the Lebesgue space $L^{d/(d-1)}(\mathbf{R}^d), d \geq 2$ with the norm bounded by a constant times $(\log N)^{1/d}$.

Key words: Kakeya problem.

1. Introduction. The purpose of this paper is to give an estimate of the Kakeya maximal function on the high dimensional Euclidean spaces.

Let f be a function on the d dimensional Euclidean space $\mathbf{R}^d, d \geq 2$. The Kakeya maximal function $K(f;x)$ is defined for a measurable function f on $\mathbf{R}^d$ by

$$K(f;x) = \sup_\theta \frac{1}{|\ell_{(x,\theta)}|} \int_{\ell_{(x,\theta)}} |f(y)|dy, \quad x \in \mathbf{R}^d,$$

where $\ell_{(x,\theta)}$ is a tube in $\mathbf{R}^d$ with the center at $x \in \mathbf{R}^d$, the diameter of bottom δ and the side length 1. δ is chosen arbitrarily but fixed, and θ implies a slope of the axis of the tube. If the direction θ is fixed, that is, independent of x, our problem is reduced to that of the Hardy-Littlewood maximal operator. In two dimensional case an estimate of $K(f;x)$ is given by A. Córdoba [2]. Some estimate for two classes of functions on the d dimensional Euclidean spaces are given by cf. e.g. S. Igari [5], and A. Carbery, E. Fernández, and F. Soria [1], but the expected L^p estimate of the Kakeya functions on the higher dimensional Euclidean spaces is not settled, cf. e.g. T. Wolff's work and N. H. Katz and J. Zahl [6].

Kakeya maximal function is deeply related to Fourier analysis of several variables and the geometry of sets in Euclidean spaces. For the detail and applications to Fourier analysis, the theory of geometry of sets and etc. see, e.g. the book of L. Grafakos [4] and its references.

2. Reduction and Theorem. In order to estimate the Kakeya maximal operator in the Euclidean space $\mathbf{R}^d (d \geq 2)$ we change the scale, so that, we assume the side length of cylinders is a positive integer $N = \delta^{-1}$ and the diameter of bottom is 1. More precisely, we define the cylinders by $\ell_{(x,\theta)} = \{y \in \mathbf{R}^d : \mathrm{dist}(L_{(x,\theta)}, y) \leq 1/2\}$ where $L_{(x,\theta)} = \{x + t\theta : -N < t < N\}$ is the axis with a slope θ. We can suppose, by rotation, θ is written by $(\theta_1, \cdots, \theta_{d-1}, 1)$.

For a measurable function f on $\mathbf{R}^d$ define $K(f;x) = N^{-1} \int \chi_{\ell_{(x,\theta)}}(y) f(y) dy$, where $\chi_{(x,\theta)}$ is the characteristic function of $\ell_{(x,\theta)}$, and θ is arbitrarily determined depending on f and x.

By the same way, for a sequence $c = \{c_a\}$ on $\mathbf{Z}^d$ put $k(c;x) = N^{-1} \sum_a c_a \chi_{(a,\theta)}(x)$. If a sequence $\{c_a\}$ is given by the characteristic function $\{\chi_P(a)\}$ of a subset P of $\mathbf{Z}^d$, then $k(c;x)$ is written as $k(P;x)$.

By a discretization argument of L^p space our estimate of $K(f)$ is equivalent to that of the operator k on the space $\ell^p(\mathbf{Z}^d)$.

Our object is to prove the following

Theorem 1. (i) *Let $d \geq 2$. The operator k is of restricted type $(d/(d-1), d/(d-1))$, that is, we have*

$$(1) \qquad \int (k(P;x))^{d/(d-1)} dx \leq C (\log N)^{1/d} P^{\#}$$

for any finite subset P of $\mathbf{Z}^d$.

(ii) *Let $p > d$. We have*

$$(2) \qquad \|K(f)\|_{L^p} \leq C(p-d)^{-1} (\log N)^{1-1/p} \|f\|_{L^p}$$

for $f \in L^p(\mathbf{R}^d)$, where C is a constant depending only on the dimention.

The second part of the theorem follows from (i) and the interpolation argument of operators, cf. e.g. Grafakos [3], p. 56.

3. Proof of the Theorem. First step. Notation and Reduction. Let us fix a positive integer

2020 Mathematics Subject Classification. Primary 42B25.

doi: 10.3792/pjaa.100.003

N. Put $\mathbf{D}=[0,N)^d\cap\mathbf{Z}^d$. To prove Theorem 1 (i) we restrict our attention to subsets P in $\mathbf{D}$. The general case is derived from this particular case.

Let P be a subset of $\mathbf{Z}^d$. For an integer $0\le g<N$ let $P_g=\{(a_1,\ldots,a_{d-1},g)\in P;(a_1,\ldots,a_{d-1},a_d)\in P\}$, which we call the g-cross section of P. For a point $x=(x_1,\cdots,x_{d-1},x_d)\in\mathbf{R}^d$ we denote $x=(\bar{x},x_d)$, where $\bar{x}=(x_1,\cdots,x_{d-1})$, and a point of the range of the function $k(P;\cdot)$ is denoted as (x,x_{d+1}) or $(\bar{x},x_d,x_{d+1})$. We assume that each point a in P is assigned an angle $\alpha=(\bar{\alpha},1),\bar{\alpha}=(\alpha_1,\ldots,\alpha_{d-1})$, which is arbitrarily given but fixed depending on a. Let μ be any integer such that $4\le\mu\le\log N/\log 2$. Put $P^\mu=\{a=(a,\alpha)\in P;2^{\mu-1}/N\le|\bar{\alpha}|<2^\mu/N\}$, where $\bar{\alpha}=(\alpha_1,\cdots,\alpha_{d-1})$. We assume $P=\bigcup_{\mu\ge 4}P^\mu$, that does not disturb generality. For the case $\mu<4$ the theorem is easily gotten. The key part of our proof is to estimate the integral $I_g^\mu=\int k(P_g^\mu;x)^{d/(d-1)}dx$, $0\le g<N$.

Second Step. We shall decompose the domain of $k(P_g^\mu;x)$ to simplify our argument. We write a point $x\in\mathbf{R}^d$ of the domain of $k(P_g^\mu)$ as $(\bar{x},x_d)$ and a point of the graph as $(\bar{x},x_d,x_{d+1})$, where $\bar{x}\in\mathbf{R}^{d-1}$, and $x_d,x_{d+1}\in\mathbf{R}$.

For $(\bar{m},m_d)\in\mathbf{Z}^{d-1}\times\mathbf{Z}$ let $J(\bar{m})=[4m_1,4m_1+1)\times\cdots\times[4m_{d-1},4(m_{d-1}+1))$, and $J^\mu(m_d)=[4m_dN/2^\mu,(4m_d+1)N/2^\mu)$ be a square of $\mathbf{R}^{d-1}$ and an interval of $\mathbf{R}$, respectively. Put $\Delta=\{(\bar{\delta},\delta_d)\in\mathbf{Z}^d;\ \bar{\delta}=(\delta_1,\cdots,\delta_{d-1}),\ 0\le\delta_i<4\ \text{ for }\ 1\le i\le d\}$, and let $J^\mu(\bar{m}+\bar{\delta},m_d+\delta_d)$ be the translation of $J^\mu(\bar{m},m_d)$ by $-(\bar{\delta},N2^{-\mu}\delta_d)$. Then $\{J^\mu(\bar{m}+\bar{\delta},m_d+\delta_d);(\bar{m},m_d)\in\mathbf{R}^d,(\bar{\delta}.\delta_d)\in\Delta\}$ is a disjoint covering of $\mathbf{R}^d$. We remark that the range of $k(P_g^\mu)$ is contained in $[-N,2N]^d\times[0,N^{-1}P_g^{\mu\#}]$. For $\delta=(\bar{\delta},\delta_d)\in\Delta$ put $\Omega^\mu(\delta)=\bigcup_m J^\mu(\bar{m}+\bar{\delta},m_d+\delta_d)$. We have $I_g^\mu=\sum_{\delta\in\Delta}I_g^\mu(\Omega^\mu(\delta))$, where

$$(3)\quad I_g^\mu(\Omega^\mu(\delta))=\sum_{\bar{m},m_d}\int_{J^\mu(\bar{m}+\bar{\delta},m_d+\delta_d)}k(P_g^\mu;x)^{d/(d-1)}dx.$$

In the following we assume $g=0$ and $\delta=(\bar{\delta},\delta_d)=0$. If it is obvious, we may omit the suffixes δ and/or g. For other cases a proof is similar. Therefore we consider the simplest case $I^\mu(\Omega^\mu(0))$. Let $b^1,b^2,\cdots,b^\Pi$, be an enumeration of points in P_0^μ, where $\Pi=P_0^{\mu\#}$. We fix it. Let us replace temporarily the tubes ℓ_{b^j} with ℓ^j defined by $\ell^1=\ell_{b^1}$ and $\ell^j=\ell_{b^j}-\ell_{b^1}-\cdots-\ell_{b^{j-1}}$ for $1<j\le\Pi$. Then $\{\ell^j\}$ is a disjoint covering of the support of $k(P_0^\mu)$. Therefore by (3) we have

$$(4)\quad I_0^\mu(\Omega^\mu(0))=\sum_{m_d}\sum_{\bar{m},j}\int_{D(j,\bar{m},m_d)}k(P_0^\mu;x)^{d/(d-1)}dx,$$

where $D(j,\bar{m},m_d)=\ell^j\cap J^\mu(\bar{m},m_d)$.

Third Step. For a subset E of $\mathbf{R}^d$ let $P_0^\mu(E)$ be the set of a in P_0^μ such that $|\ell_a\cap E|>0$.

Pick any point b^j in P_0^μ. Choose any point $(\bar{m},m_d)$ in $\mathbf{D}^d$ such that $|\ell_{b^j}\cap J^\mu(\bar{m},m_d)|>0$. For each b^j there exist at most $[N/4]$ such points $\bar{m}$, say $\bar{m}(b^j)$, and furthermore, for each $\bar{m}(b^j)$ there corresponds uniquely an integer $m_d=m_d(b^j)$. Conversely, for a given $m_d(b^j)$ there corresponds uniquely $\bar{m}$ such that $\bar{m}=\bar{m}(b^j)$. In this way we get a sequence $(\bar{m}(b^j),m_d(b^j))$ in $\mathbf{Z}^d$ for each $j=1,2,\cdots,\Pi$.

We write $(\bar{m}(b^j),m_d(b^j))=(\bar{m}(j),m_d(j))$ for simplicity. Then by our definition

$$(5)\qquad k(P_0^\mu;x)\le N^{-1}P_0^\mu(\bar{m}(j),m_d(j))^\#$$

for almost all x in $\ell^j\cap J^\mu(\bar{m}(j),m_d(j))$. For $m(j)=(\bar{m}(j),m_d(j))$ let $I(m(j))$ be the set of i's such that $|\ell^i\cap J^\mu(\bar{m}(j),m_d(j))|>0$. The (5) holds also for almost all x in $\ell^i\cap J^\mu(\bar{m}(j),m_d(j))$.

Since $\{\ell^i,i\in I(m(j))\}$ is a disjoint family. Applying (5) to (4), we have

$$(6)\ I_0^\mu(\Omega^\mu(0))\le\sum_{(\bar{m},m_d)}\sum_j\sum_{i\in I(m(j))}\int_{D(i,m(j),m_d(j))}k(P_0^\mu;x)^{d/(d-1)}dx.$$

Fourth Step. To estimate the right hand side of the last inequality we use the following lemma.

Lemma 1. *Let $1\le\mu\le[\log N/\log 2]+1$ be an integer. Then the family $\{P_0^\mu(\bar{m}(j),m_d(j))\}$ has the following properties*:

(i) $\bigcup_{j=1}^{\Pi}\bigcup_{(\bar{m},m_d)}P_0^\mu(\bar{m}(j),m_d(j))=P_0^\mu(\Omega^\mu(0))$,

(ii) *If $m_d=m_d(i)=m_d(j)$, then $P_0^\mu(\bar{m}(b^i),m_d(i))$ and $P_0^\mu(\bar{m}(j),m_d(b^j))$ are either identical or mutually disjoint,*

(iii) *For each m_d we have*

$$\sum_{\bar{m}}\sum_{i\in I(m(j))}\int_{D(i,\bar{m}(j),m_d)}((k(\bar{m}(j),m_d(j))^{d/(d-1)}dx\le N^{-d/(d-1)}2^\mu P_0^{\mu\#}.$$

Proof of Lemma. (i) By definition, a point b^j belongs to $P_0^\mu(\Omega^\mu(0))$ if and only if there exists a point $(\bar{m},m_d)$, such that $|\ell_{b^j}\cap J^\mu(\bar{m},m_d)|>0$. Thus the right hand side of (i) is contained in the left hand side. The converse is obvious.

(ii) follows immediately from the definition of $P_0^\mu(\bar{m}(i),m_d(i))$.

(iii) Fix m_d. For $1 \leq j \leq \Pi$ put $P_0^\mu(j, m_d) = \{b^i; m_d(b^i) = m_d(b^j)\}$.
We have the following
(a) for each m_d $\{P_0^\mu(j, m_d); 1 \leq j \leq \Pi\}$ are mutually disjoint.
(b) $P_0^\mu(j, m_d)$ is contained in the disk of $\mathbf{D}_0$ with the center at $(\bar{m}(j), 0)$ and the radius bounded by $(4m_d(j)+1)N/2^\mu \times (2^\mu/N) = 4m_d(j)+1 \leq 2^\mu$.
(c) $\{\ell^i \cap D(j, m(j), m_d); b^i \in P_0^\mu(j, m_d)\}$ is a disjoint covering of $D(j, m(j), m_d)$.
(d) $P_0^\mu(m_d)^{\# d/(d-1)} \leq 2^\mu B_d^{1/(d-1)} P_0^\mu(\bar{m}(j), m_d(j))^{\#}$, where B_{d-1} is the volume of the unit ball of $d-1$ dimensional Euclidean space.

(a), (b) and (c) follow easily from our definitions. To prove (iii) note that $P_0^\mu(\bar{m}(j), m_d(j))^{\# d/(d-1)} \leq P_0^\mu(\bar{m}(j), m_d(j))^{\#} \cdot (B_{d-1}(2^\mu))^{1/(d-1)} \leq ((B_{d-1}2^\mu)^{d-1})^{1/(d-1)} P_0^\mu(\bar{m}(j), m_d(j))^{\#}$. Thus $\sum_{\bar{m}} P_0^\mu(\bar{m}(j), m_d(j))^{\# d/(d-1)} \leq 2^\mu N^{-d/(d-1)} \sum_{\bar{m}} P_0^\mu(\bar{m}(j), m_d(j))^{\#}$.

Thus the left hand side of the inequality (d) is up to a constant times bounded by

$$2^\mu N^{-d/(d-1)} \sum_{\bar{m}} \sum_{I(j)} \sum_{i \in P_0^\mu(j, m_d)} \int_{\ell^i \cap D(j, \bar{m}(j), m_d)} (P_0^\mu(\bar{m}(j), m_d))^{\#} dx.$$

If $i, i' \in P_0(j, m_d), i \neq i'$, the corresponding integral domains are disjoint but integrands remain the same value. Therefore the last expression does not exeed

$$2^\mu N^{-d/(d-1)} \sum_{\bar{m}} \sum_{I(j)} \int_{J^\mu(\bar{m}(j).m_d(j))} P_0^\mu(\bar{m}(j), m_d(j))^{\#} dx \leq 2^\mu N^{-d/(d-1)} P_0^{\mu\#},$$

since $P_0^\mu(\bar{m}(j), m_d(j)), j = 1, 2, \ldots, \Pi$, are mutually disjoint. □

Applying Lemma (iii) to inequality (6) we get $I_0^\mu(\Omega^\mu(0)) \leq N^{-1/(d-1)} P_0^{\mu\#}$, which implies that the last inequality holds good for subsets $P_g^\mu, 0 < g \leq N$, and for the integral domain $\Omega(\bar{\delta}, \delta_d), \delta = (\bar{\delta}, \delta_d) \neq 0$ in place of $\Omega^\mu(0)$ by translations. Thus we get

$$I_g^\mu = \sum_{\delta \in \Delta} I_g^\mu(\Omega(\bar{\delta}, \delta_d)) \leq 4^d N^{-1/(d-1)} P_g^{\mu\#} \tag{7}$$

for all $P \subset \mathbf{D}$ and $0 \leq g < N$.

Fifth Step. Recall the definition of the $g-$cross section in section 3 and the corresponding angles of the tubes. We shall remove the the restrictions of the functional $k(\cdot; x)$ on sequences $\{\chi_P\}, P \subset \mathbf{R}_d$, and on the angles.

Let $P \subset \mathbf{D}$ and $P^\mu = \bigcup_{0 \leq g < N} P_g^\mu$. By the Hőlder inequality with indices $1/d + (d-1)/d = 1$, we have $k(P^\mu; x)^{d/(d-1)} \leq N^{1/(d-1)} \sum_g (k(P_g^\mu; x))^{d/(d-1)}$. Thus we have by (7)

$$\begin{aligned} I^\mu &= \int_{\mathbf{R}^d} k(P^\mu; x)^{d/(d-1)} dx \\ &\leq N^{1/(d-1)} \sum_g \int_{\mathbf{R}^d} (k(P_g^\mu; x))^{d/(d-1)} dx \\ &\leq 4^d \sum_g P_g^{\mu\#} = 4^d P^{\mu\#}. \end{aligned} \tag{8}$$

To remove the restriction for angles put $P'' = P - P'$, where $P' = \bigcup_{\mu=3}^{1+[\log N/\log 2]} P^\mu$. When $\mu < 3$, every tube ℓ_a is contained in a rectangle $[-4, 4]^{d-1} \times [N, N] + a = R_a$. Thus $|\ell_a|^{-1} \int_{\ell_a} |f(y)| dy \leq 8^{d-1} |R_a|^{-1} \int_{R_a} |f(y)| dy$ for all $a \in P''$. Thus we have $\int (k(P''))^{d/(d-1)} dx \lesssim P''^{\#}$ by a well known calculus or by an application of Hardy-Littlewood maximal inequality.

Therefore our main concern is in the subset P'. Applying the Hőlder inequality again for the sum $\sum_\mu$ over $\{\mu; 3 \leq \mu \leq 1 + [\log N/\log 2]\}$ with indices $1/d + (d-1)/d = 1$ we have

$$\begin{aligned} I' &= \int_{\mathbf{R}^d} (k(P'; x))^{d/(d-1)} dx \\ &= \int_{\mathbf{R}^d} \Big(k(\sum_{\{\mu\}} P^\mu; x)\Big)^{d/(d-1)} dx \\ &\leq (1 + [\log N/\log 2])^{1/(d-1)} \sum_{\mu \geq 3} \int k(P^\mu; x)^{d/(d-1)} dx. \end{aligned}$$

Due to (8) we get

$$I' \leqq c_d (\log N)^{1/(d-1)} P^{\#} \tag{9}$$

for all $P \subset \mathbf{D}$, where c_d is a constant depending only on d but may be defferent each occasion. Therefore we get

$$I = \int_{\mathbf{R}^d} (k(P; x))^{d/(d-1)} dx \leq c_d (\log N)^{1/(d-1)} P^{\#} \tag{10}$$

for all $P \subset \mathbf{D}$.

Put $\mathbf{D}(m) = \mathbf{D} + 2Nm$ for $m \in \mathbf{Z}^d$ and $\mathbf{Z}^d(i) = \bigcup_m \mathbf{D}(m) + N\varepsilon^i$, where $\varepsilon^i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$, $i = 1, \cdots, d$. If $m \neq m'$, the ranges of $k(\mathbf{D}(m))$ and $k(\mathbf{D}(m'))$ are disjoint. Thus (10) holds for $\mathbf{D}(m) \cup \mathbf{D}(m')$ with the same constant c_d. This implies that (10) holds for all subets of $\mathbf{Z}^d(i)$, and thus for every subset of $\mathbf{Z}^d$ with constant $2^d c_d$. □

Acknowledgements. I wish to express my gratitude to Profs. Akihiko Miyachi and Hitoshi

Tanaka for their cooperation and encouragement. In particular, Prof. Miyachi gave helpful advices.

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