

# Log canonical inversion of adjunction

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**Abstract:** This is a short note on the log canonical inversion of adjunction.

**Key words:** Inversion of adjunction; adjunction; log canonical singularities; complex analytic spaces.

**1. Introduction.** The following theorem is Kawakita's inversion of adjunction on log canonicity (see [8, Theorem]). Although [8, Theorem] is formulated and proved only for algebraic varieties, his clever and mysterious proof in [8] works in the complex analytic setting. Here we will prove it as an application of the minimal model theory for projective morphisms of complex analytic spaces established in [6] following the argument in [7] with some suitable modifications. Our proof is more geometric than Kawakita's.

**Theorem 1.1** (Log canonical inversion of adjunction, see [8, Theorem]). *Let  $X$  be a normal complex variety and let  $S + B$  be an effective  $\mathbf{R}$ -divisor on  $X$  such that  $K_X + S + B$  is  $\mathbf{R}$ -Cartier,  $S$  is reduced, and  $S$  and  $B$  have no common irreducible components. Let  $\nu: S^\nu \rightarrow S$  be the normalization with  $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$ , where  $B_{S^\nu}$  denotes Shokurov's different. Then  $(X, S + B)$  is log canonical in a neighborhood of  $S$  if and only if  $(S^\nu, B_{S^\nu})$  is log canonical.*

We note that  $X$  is not necessarily an algebraic variety in Theorem 1.1. It is only a complex analytic space. In this note, we will freely use [6] and [2]. We assume that the reader is familiar with the basic definitions and results of the minimal model theory for algebraic varieties (see, for example, [3–5, 9], and so on).

## 2. Quick review of the analytic MMP.

In this section, we quickly explain the minimal model theory for projective morphisms between complex analytic spaces established in [6].

**2.1** (Singularities of pairs). As in the algebraic case, we can define *kawamata log terminal*

*pairs, log canonical pairs, purely log terminal pairs, divisorial log terminal pairs*, and so on, for complex analytic spaces. For the details, see [6, Section 3].

One of the main contributions of [6] is to find out a suitable complex analytic formulation in order to make the original proof of [3] work with only some minor modifications.

**2.2.** Let  $\pi: X \rightarrow Y$  be a projective morphism between complex analytic spaces. A compact subset of an analytic space is said to be *Stein compact* if it admits a fundamental system of Stein open neighborhoods. It is well known that if  $W$  is a Stein compact semianalytic subset of  $Y$  then  $\Gamma(W, \mathcal{O}_Y)$  is noetherian. From now on, we fix a Stein compact subset  $W$  of  $Y$  such that  $\Gamma(W, \mathcal{O}_Y)$  is noetherian. Then we can formulate and prove the cone and contraction theorem over some open neighborhood of  $W$  as in the usual algebraic case. This is essentially due to Nakayama (see [10]). We say that  $X$  is  $\mathbf{Q}$ -factorial over  $W$  if every prime divisor defined on an open neighborhood of  $\pi^{-1}(W)$  is  $\mathbf{Q}$ -Cartier at any point  $x \in \pi^{-1}(W)$ . Then, in [6], we show that we can translate almost all the results in [3] into the above analytic setting suitably (see [6, Section 1]).

Hence we have the minimal model program with ample scaling as in the algebraic case. In Section 4, we will use it in the proof of Theorem 1.1.

**2.3** (Minimal model program with ample scaling). Let  $(X, \Delta)$  be a divisorial log terminal pair such that  $X$  is  $\mathbf{Q}$ -factorial over  $W$  and let  $C \geq 0$  be a  $\pi$ -ample  $\mathbf{R}$ -divisor on  $X$  such that  $(X, \Delta + C)$  is log canonical and that  $K_X + \Delta + C$  is nef over  $W$ . Then we can run the  $(K_X + \Delta)$ -minimal model program with scaling of  $C$  over  $Y$  around  $W$  from

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$(X_0, \Delta_0) := (X, \Delta)$  as in the algebraic case. We put  $C_0 := C$ . Thus we get a sequence of flips and divisorial contractions

$$(X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \\ \xrightarrow{\phi_{i-1}} (X_i, \Delta_i) \xrightarrow{\phi_i} (X_{i+1}, \Delta_{i+1}) \xrightarrow{\phi_{i+1}} \cdots$$

over  $Y$  with  $\Delta_i := (\phi_{i-1})_* \Delta_{i-1}$  and  $C_i := (\phi_{i-1})_* C_{i-1}$  for every  $i \geq 1$ . We note that each step  $\phi_i$  exists only after shrinking  $Y$  around  $W$  suitably. We also note that

$$\lambda_i := \inf\{\mu \in \mathbf{R}_{\geq 0} \mid K_{X_i} + \Delta_i + \mu C_i \text{ is nef over } W\}$$

and that each step  $\phi_i$  is induced by a  $(K_{X_i} + \Delta_i)$ -negative extremal ray  $R_i$  such that  $(K_{X_i} + \Delta_i + \lambda_i C_i) \cdot R_i = 0$ . We have

$$\lambda_{-1} := 1 \geq \lambda_0 \geq \lambda_1 \geq \cdots$$

such that this sequence is

- finite with  $\lambda_{N-1} > \lambda_N = 0$ , or
- infinite with  $\lim_{i \rightarrow \infty} \lambda_i = 0$ .

Of course, it is conjectured that the above minimal model program always terminates after finitely many steps. Unfortunately, however, it is still widely open even when  $\pi: X \rightarrow Y$  is algebraic.

Anyway, for the details of the minimal model theory for projective morphisms of complex analytic spaces, see [6].

**3. Zariski's subspace theorem.** In this short section, we quickly review *Zariski's subspace theorem* following [1].

**3.1** (see [1, (1.1)]). Let  $R_1$  and  $R_2$  be noetherian local rings. Then we say that  $R_2$  *dominates*  $R_1$  if  $R_1$  is a subring of  $R_2$  and  $m_{R_1} \subset m_{R_2}$  holds, where  $m_{R_1}$  (resp.  $m_{R_2}$ ) is the maximal ideal of  $R_1$  (resp.  $R_2$ ).

**3.2** (see [1, (1.1)]). Let  $R_1$  and  $R_2$  be noetherian local rings such that  $R_1$  is a subring of  $R_2$ . We say that  $R_1$  is a *subspace* of  $R_2$  if  $R_1$  with its Krull topology is a subspace of  $R_2$  with its Krull topology. This means that  $R_2$  dominates  $R_1$  and there exists a sequence of non-negative integers  $a(n)$  such that  $a(n)$  tends to infinity with  $n$  and  $R_1 \cap m_{R_2}^n \subset m_{R_1}^{a(n)}$  holds for every  $n \geq 0$ .

**3.3** (see [1, (1.1)]). Let  $R_1$  and  $R_2$  be noetherian local domains such that  $R_1$  is a subring of  $R_2$ . Then  $\text{trdeg}_{R_1} R_2$  denotes the *transcendence degree* of the quotient field of  $R_2$  over the quotient field of  $R_1$ . Let  $h: R_2 \rightarrow R_2/m_{R_2}$  be the canonical surjection, where  $m_{R_2}$  is the maximal ideal of  $R_2$ . Let  $k$  be the

quotient field of  $h(R_1)$  in  $h(R_2)$ . Then  $\text{trdeg}_k h(R_2)$  is called the *residual transcendence degree* of  $R_2$  over  $R_1$  and is denoted by  $\text{restrdeg}_{R_1} R_2$ .

We need the following form of Zariski's subspace theorem.

**Theorem 3.4** (see, for example, [1, (10.13)]). *Let  $R_1$  and  $R_2$  be noetherian local domains such that  $R_1$  is analytically irreducible,  $R_2$  dominates  $R_1$ ,  $\text{trdeg}_{R_1} R_2 < \infty$ , and  $\dim R_1 + \text{trdeg}_{R_1} R_2 = \dim R_2 + \text{restrdeg}_{R_1} R_2$ . Then  $R_1$  is a subspace of  $R_2$ .*

Here we do not prove Theorem 3.4. For the details, see [1, §10].

**4. Proof of Theorem 1.1.** Let us prove Theorem 1.1 following the argument in [7], where the log canonical inversion of adjunction was established for log canonical centers of arbitrary dimension. Our proof given below uses Zariski's subspace theorem as in [8].

Proof of Theorem 1.1. In this proof, we will closely follow the argument in [7] with some suitable modifications. If  $(X, S + B)$  is log canonical in a neighborhood of  $S$ , then it is easy to see that  $(S^\nu, B_{S^\nu})$  is log canonical by adjunction. Therefore, it is sufficient to prove that  $(X, S + B)$  is log canonical near  $S$  under the assumption that  $(S^\nu, B_{S^\nu})$  is log canonical. Without loss of generality, we may assume that  $S$  is irreducible. We take an arbitrary point  $P \in S$ . We can replace  $X$  with a relatively compact Stein open neighborhood of  $P$  since the statement is local. From now on, we will freely shrink  $X$  around  $P$  suitably throughout the proof without mentioning it explicitly.

**Step 1.** In this step, we will see that we can reduce the problem to the case where  $K_X + S + B$  is  $\mathbf{Q}$ -Cartier.

The argument here is more or less well known to the experts and is standard in the theory of minimal models. Hence we will only give a sketch of the proof. As usual, we can write

$$K_X + S + B = \sum_{p=1}^q r_p (K_X + S + B_p)$$

such that  $K_X + S + B_p$  is  $\mathbf{Q}$ -Cartier,  $0 < r_p < 1$  for every  $p$  with  $\sum_{p=1}^q r_p = 1$ , and  $(S^\nu, B_p^\nu)$  is log canonical for every  $p$ , where  $K_{S^\nu} + B_p^\nu = \nu^*(K_X + S + B_p)$ . Note that if  $(X, S + B_p)$  is log canonical near  $S$  for every  $p$  then  $(X, S + B)$  is log canonical in a suitable neighborhood of  $S$ . Therefore, we can replace  $(X, S + B)$  with  $(X, S + B_p)$  and assume

that  $K_X + S + B$  is  $\mathbf{Q}$ -Cartier. This is what we wanted.

**Step 2.** In this step, we will make a good partial resolution of singularities of the pair  $(X, S + B)$  by using the minimal model program established in [6] (see also Section 2).

Let  $W$  be a Stein compact subset of  $X$  such that  $\Gamma(W, \mathcal{O}_X)$  is noetherian and that  $W$  contains some open neighborhood of  $P$ . By [6, Theorem 1.21], we can take a projective bimeromorphic morphism  $\mu: Y \rightarrow X$  with  $K_Y + \Delta_Y = \mu^*(K_X + S + B)$  such that

- (i)  $Y$  is  $\mathbf{Q}$ -factorial over  $W$ ,
- (ii)  $\Delta_Y$  is effective and  $\Delta_Y = \sum_j d_j \Delta_j$  is the irreducible decomposition,
- (iii) the pair

$$\left( Y, \Delta'_Y := \sum_{d_j \leq 1} d_j \Delta_j + \sum_{d_j > 1} \Delta_j \right)$$

is divisorial log terminal, and

- (iv) every  $\mu$ -exceptional divisor appears in  $(\Delta'_Y)^{-1} := \sum_{d_j \geq 1} \Delta_j$ .

Note that  $\mu: Y \rightarrow X$  is sometimes called a dlt blow-up of  $(X, S + B)$  in the literature (see [6, Theorem 1.21]). We write  $\Delta'_Y = T + \Gamma$ , where  $T$  is the strict transform of  $S$  and  $\Gamma := \Delta'_Y - T$ , and put

$$\Sigma := \Delta_Y - T - \Gamma = \Delta_Y - \Delta'_Y.$$

We take an effective Cartier divisor  $E$  on  $Y$  such that  $-E$  is  $\mu$ -ample and  $K_Y + T + \Gamma - E$  is  $\mu$ -nef over  $W$ . We note that we can choose  $E$  such that  $E$  and  $T$  have no common components. Then we run the  $(K_Y + T + \Gamma)$ -minimal model program with scaling of  $-E$  over  $X$  around  $W$ . We obtain a sequence of flips and divisorial contractions:

$$\begin{aligned} (Y, T + \Gamma) &= (Y_0, T_0 + \Gamma_0) \xrightarrow{\phi_0} (Y_1, T_1 + \Gamma_1) \\ &\xrightarrow{\phi_1} (Y_2, T_2 + \Gamma_2) \xrightarrow{\phi_2} \cdots \\ &\xrightarrow{\phi_{i-1}} (Y_i, T_i + \Gamma_i) \xrightarrow{\phi_i} \cdots \end{aligned}$$

Note that each step exists only after shrinking  $X$  around  $W$  suitably. Let  $\mu_i: Y_i \rightarrow X$  be the induced morphism. For any divisor  $G$  on  $Y$ , we let  $G_i$  denote the pushforward of  $G$  on  $Y_i$ . We put  $\lambda_{-1} := 1$ . By construction, there exists a non-increasing sequence of rational numbers  $\lambda_i \geq \lambda_{i+1}$  with  $i \geq 0$  that is either

- finite with  $\lambda_{N-1} > \lambda_N = 0$ , or
- infinite with  $\lim_{i \rightarrow \infty} \lambda_i = 0$

such that  $K_{Y_i} + T_i + \Gamma_i - \lambda E_i$  is nef over  $W$  for all  $\lambda_{i-1} \geq \lambda \geq \lambda_i$ . Without loss of generality, we may assume that each  $\phi_i$  is a flip for every  $i \geq i_0$  or that  $i_0 = N$ , that is, the minimal model program stops at  $i_0 = N$ . For any positive rational number  $t$ , there is an effective  $\mathbf{Q}$ -divisor  $\Theta_t$  on  $Y$  such that  $\Theta_t \sim_{\mathbf{Q}} \Gamma - tE$  and  $(Y, T + \Theta_t)$  is purely log terminal with  $[T + \Theta_t] = T$ . In this case, we see that if  $t < \lambda_{i-1}$  then  $(Y_i, T_i + \Theta_{t,i})$  is purely log terminal. In particular,  $(Y_i, \Theta_{t,i})$  is kawamata log terminal.

**Step 3.** In this step, we will check that  $T_i \cap \Sigma_i = \emptyset$  holds for every  $i$ .

We note that  $T_i$  is normal since  $(Y_i, T_i + \Gamma_i)$  is a divisorial log terminal pair. Therefore,  $\mu_i: T_i \rightarrow S$  factors through  $\nu: S^\nu \rightarrow S$ . By construction, we have  $K_{Y_i} + T_i + \Gamma_i + \Sigma_i = \mu_i^*(K_X + S + B)$ . Hence

$$(4.1) \quad K_{T_i} + \text{Diff}_{T_i}(\Gamma_i + \Sigma_i) := (K_{Y_i} + T_i + \Gamma_i + \Sigma_i)|_{T_i} = (\mu'_i)^*(K_{S^\nu} + B_{S^\nu})$$

holds, where  $\mu'_i: T_i \rightarrow S^\nu$ . Assume that  $T_i \cap \Sigma_i$  is not empty. Then we see that  $(T_i, \text{Diff}_{T_i}(\Gamma_i + \Sigma_i))$  is not log canonical. By (4.1), this is a contradiction since  $(S^\nu, B_{S^\nu})$  is log canonical by assumption. This implies that  $T_i \cap \Sigma_i = \emptyset$  holds for every  $i$ . In particular, we have

$$\begin{aligned} K_{T_i} + \text{Diff}_{T_i}(\Gamma_i + \Sigma_i) &= (K_{Y_i} + T_i + \Gamma_i + \Sigma_i)|_{T_i} \\ &= (K_{Y_i} + T_i + \Gamma_i)|_{T_i} \\ &=: K_{T_i} + \text{Diff}_{T_i}(\Gamma_i). \end{aligned}$$

**Step 4.** In this step, we will show that  $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$  is an isomorphism for every  $i$ . Moreover, we will prove that if  $\phi_i$  is a flip then  $\phi_i$  is an isomorphism on some open neighborhood of  $T_i$ .

First, we assume that  $\phi_i$  is a flip. We consider the following flipping diagram

$$\begin{array}{ccc} (Y_i, T_i + \Gamma_i) & \xrightarrow{\phi_i} & (Y_{i+1}, T_{i+1} + \Gamma_{i+1}) \\ & \searrow \varphi_i & \swarrow \varphi_i^+ \\ & & Z_i \end{array}$$

and we let  $W_i$  denote the normalization of  $\varphi_i(T_i)$ . Let  $C$  be any flipping curve. If  $C$  is contained in  $T_i$ , then we obtain

$$(4.2) \quad \begin{aligned} (K_{Y_i} + T_i + \Gamma_i) \cdot C \\ = (K_{Y_i} + T_i + \Gamma_i + \Sigma_i) \cdot C = 0 \end{aligned}$$

since  $T_i \cap \Sigma_i = \emptyset$  by Step 3. This is absurd. Hence this implies that the natural map  $T_i \rightarrow W_i$  is an isomorphism. By the same argument, we see that the natural map  $T_{i+1} \rightarrow W_i$  is also an isomorphism.

This means that  $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$  is an isomorphism when  $\phi_i$  is a flip. By the above argument, we see that  $T_{i+1}$  (resp.  $T_i$ ) does not contain any flipped (resp. flipping) curves. Note that if  $T_i \cdot C > 0$  holds for some flipping curve  $C$  then  $-T_{i+1}$  is  $\varphi_i^+$ -ample. Hence  $T_i$  is disjoint from the flipping locus. This implies that  $\phi_i$  is an isomorphism near  $T_i$  when  $\phi_i$  is a flip.

Next, we assume that  $\phi_i$  is a divisorial contraction. In this case,  $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$  is obviously a projective bimeromorphic morphism between normal complex varieties. Let  $C$  be any curve contracted by  $\phi_i$ . Assume that  $C$  is contained in  $T_i$ . Then, by the same computation as in (4.2), we get a contradiction. This means that  $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$  does not contract any curves. Thus,  $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$  is an isomorphism.

We get the desired statement.

**Step 5.** In this step, we will prove that the natural restriction map

$$(\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0}) \rightarrow (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

is surjective over some open neighborhood of  $P$  for every positive integer  $m \geq a/\lambda_{i_0-1}$  such that  $m\Sigma$  is an integral divisor, where  $a$  is the smallest positive integer such that  $aE_{i_0}$  is Cartier.

By definition,  $aE_{i_0}$  is Cartier. By Step 4,  $Y_{i_0} \dashrightarrow Y_i$  is an isomorphism on some open neighborhood of  $T_{i_0}$  for every  $i \geq i_0$ . Therefore,  $aE_i$  is Cartier on some open neighborhood of  $T_i$  for every  $i \geq i_0$ . Since  $(Y_i, T_i + \Gamma_i)$  is divisorial log terminal and  $T_i$  is a  $\mathbf{Q}$ -Cartier integral divisor, we have the following short exact sequence:

$$(4.3) \quad 0 \rightarrow \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i - T_i) \rightarrow \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) \rightarrow \mathcal{O}_{T_i}(-aE_i) \rightarrow 0$$

for every  $i \geq i_0$  and every  $m$  such that  $m\Sigma_i$  is integral (cf. [9, Proposition 5.26]). Here, we used the fact that  $T_i \cap \Sigma_i = \emptyset$  (see Step 3). Let  $U$  be an open neighborhood of  $P$  contained in  $W$ . For every positive integer  $m \geq a$  such that  $m\Sigma$  is an integral divisor, there exists  $i$  such that  $\lambda_{i-1} \geq a/m \geq \lambda_i$ . If further  $m \geq a/\lambda_{i_0-1}$ , then  $i \geq i_0$ . Since

$$\begin{aligned} & -m\Sigma_i - aE_i - T_i - (K_{Y_i} + \Theta_{\frac{a}{m}, i}) \\ & \sim_{\mathbf{Q}, \mu_i} (m-1) \left( K_{Y_i} + T_i + \Gamma_i - \frac{a}{m} E_i \right), \end{aligned}$$

$(Y_i, \Theta_{\frac{a}{m}, i})$  is kawamata log terminal,  $K_{Y_i} + T_i + \Gamma_i - \frac{a}{m} E_i$  is nef over  $U$ , we obtain that

$$(4.4) \quad R^1(\mu_i)_* \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i - T_i) = 0$$

on  $U$  by the Kawamata–Viehweg vanishing theorem for projective bimeromorphic morphisms of complex analytic spaces. Hence the natural restriction map

$$\begin{aligned} (\mu_i)_* \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) & \rightarrow (\mu_i)_* \mathcal{O}_{T_i}(-m\Sigma_i - aE_i) \\ & = (\mu_i)_* \mathcal{O}_{T_i}(-aE_i) \end{aligned}$$

is surjective on  $U$  by (4.3) and (4.4). Note that

$$\begin{aligned} (\mu_i)_* \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) & \\ & = (\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0}) \end{aligned}$$

and

$$(\mu_i)_* \mathcal{O}_{T_i}(-aE_i) = (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

hold because  $Y_{i_0} \dashrightarrow Y_i$  is an isomorphism in codimension one and  $Y_{i_0} \dashrightarrow Y_i$  is an isomorphism on some open neighborhood of  $T_{i_0}$  by Step 4, respectively. Thus, the natural restriction map

$$(4.5) \quad (\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0}) \rightarrow (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

is surjective on  $U$  for every positive integer  $m \geq a/\lambda_{i_0-1}$  such that  $m\Sigma$  is an integral divisor. This is what we wanted.

**Step 6.** In this final step, we will get a contradiction by assuming that  $(X, S+B)$  is not log canonical at  $P$ . Here, we will use Zariski's subspace theorem as in [8].

The assumption implies that  $P \in \mu(\Sigma)$ . Note that the non-log canonical locus of  $(X, S+B)$  is  $\mu(\Sigma)$  set theoretically. By construction,  $(Y_i, T_i + \Gamma_i)$  is divisorial log terminal. Therefore, the non-log canonical locus of  $(Y_i, T_i + \Gamma_i + \Sigma_i)$  is nothing but the support of  $\Sigma_i$ . Therefore,  $\mu(\Sigma) = \mu_i(\Sigma_i)$  holds set theoretically for every  $i$ . Hence we have  $P \in \mu_{i_0}(\Sigma_{i_0})$ .

**Claim.** *Let  $\mathcal{O}_{X,P}$  be the localization of  $\mathcal{O}_X$  at  $P$  and let  $m_P$  denote the maximal ideal of  $\mathcal{O}_{X,P}$ . For every positive integer  $n$ , there exists a divisible positive integer  $\nu(n)$  such that*

$$(\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-\nu(n)\Sigma_{i_0} - aE_{i_0})_P \subset m_P^n \subset \mathcal{O}_{X,P}$$

holds, where  $(\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-\nu(n)\Sigma_{i_0} - aE_{i_0})_P$  denotes the localization of  $(\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-\nu(n)\Sigma_{i_0} - aE_{i_0})$  at  $P$ .

**Proof of Claim.** We take  $Q \in \Sigma_{i_0}$  such that  $\mu_{i_0}(Q) = P$ . We consider  $\mathcal{O}_{X,P} \hookrightarrow \mathcal{O}_{Y_{i_0},Q}$ , where  $\mathcal{O}_{Y_{i_0},Q}$  is the localization of  $\mathcal{O}_{Y_{i_0}}$  at  $Q$ . It is well known that  $\mathcal{O}_{X,P}$  is excellent. Therefore,  $\mathcal{O}_{X,P}$  is analytically irreducible since  $X$  is normal. Since

$\mu_{i_0}: Y_{i_0} \rightarrow X$  is a projective bimeromorphic morphism, the quotient field of  $\mathcal{O}_{Y_{i_0}, Q}$  coincides with the one of  $\mathcal{O}_{X, P}$ . We note that the natural map  $\mathcal{O}_{X, P} \rightarrow \mathcal{O}_{Y_{i_0}, Q}/m_Q$  is surjective, where  $m_Q$  is the maximal ideal of  $\mathcal{O}_{Y_{i_0}, Q}$ . Hence we can use Zariski's subspace theorem (see Theorem 3.4). Thus we get a large and divisible positive integer  $\nu(n)$  with the desired property.  $\square$

We consider the localization of the following restriction map  $\mathcal{O}_X \simeq (\mu_{i_0})_* \mathcal{O}_{Y_{i_0}} \rightarrow (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}$  at  $P$ . We put  $A = \mathcal{O}_{X, P}$ ,  $M = ((\mu_{i_0})_* \mathcal{O}_{T_{i_0}})_P$ , and  $N = ((\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0}))_P$ . Then, by the surjection (4.5) in Step 5 and Claim, we obtain that  $N = (0)$  by Lemma 4.1 below. This is a contradiction.

Hence, we obtain that  $(X, S + B)$  is log canonical at  $P$ . Since  $P$  is an arbitrary point of  $S$ ,  $(X, S + B)$  is log canonical in a neighborhood of  $S$ . We finish the proof of Theorem 1.1.  $\square$

We used the following easy commutative algebra lemma in the above proof of Theorem 1.1.

**Lemma 4.1.** *Let  $(A, \mathfrak{m})$  be a noetherian local ring, let  $M$  be a finitely generated  $A$ -module, and let  $\varphi: A \rightarrow M$  be a homomorphism of  $A$ -modules. Let  $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$  be a chain of ideals of  $A$  such that there exists  $\nu(n)$  satisfying  $I_{\nu(n)} \subset \mathfrak{m}^n$  for every positive integer  $n$ . Let  $N$  be an  $A$ -submodule of  $M$ . Assume that  $\varphi(I_k) = N$  holds for every positive integer  $k$ . Then we have  $N = (0)$ .*

*Proof.* Let  $b$  be any element of  $N$ . Then we can take  $a \in I_{\nu(n)} \subset \mathfrak{m}^n$  such that  $\varphi(a) = b$ . This implies that  $b = \varphi(a) \in \mathfrak{m}^n M$ . Hence  $b \in \mathfrak{m}^n M$  holds for every positive integer  $n$ . Thus we obtain  $b \in \bigcap_n \mathfrak{m}^n M = (0)$ . Therefore,  $b = 0$  holds, that is,  $N = (0)$ .  $\square$

We close this short note with a remark.

**Remark 4.2.** If  $(X, S + B)$  is algebraic in Theorem 1.1, then we do not need [6]. It is sufficient to use the minimal model program at the level of [3], the well-known relative Kawamata–Viehweg vanishing theorem, and Zariski's subspace theorem (see, for example, [1, (10.6)]). Our proof given here is longer than Kawakita's one (see [8]). However, it looks more accessible for the experts of the minimal

model program since the argument is more or less standard.

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