55. *Lie derivatives in general space of paths.**

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§ 0. *Introduction.* In a series of Notes published in these Proceedings,[1] the present author has studied the infinitesimal deformations in affinely connected spaces. The main purpose of the present Note is to study the infinitesimal deformations in the general space of paths and to generalise some of the results obtained in the above cited Notes. To my knowledge, there are only three parers on the infinitesimal deformations in the generalised spaces, the papers by M.S. Knebelman,[2] by S. Hokari[3] and by E. T. Davies.[4]

In Paragraph 1, we expose some formulae in the geometry of general space of paths which will be useful later. In the next Paragraph, we study the Lie derivations of tensors whose components are functions not only of position but also of direction. In Paragraph 3, we shall define the Lie derivatives of the affine connection and study some of its fundamental properties.

In Paragraph 4, we study the affine and projective collineations in the general space of paths which were also studied by M.S. Knebelman.

In the last Paragraph, we define the deformed space whose components of the affine connection are $\Gamma^{\lambda}_{\mu\nu}+D\Gamma^{\lambda}_{\mu\nu}$, and calculate the curvature tensor of the deformed space. The full detail will be published elsewhere.

§ 1. *General space of paths.*[5] A general space of paths is an n-dimensional space in which is given a system of curves, called paths, such that through any two points given in a properly restricted region, there passes one and only one path. If we introduce, in this general space of paths, a system of coordinates

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1) K. Yano: Bemerkungen über infinitesimale Deformationen eines Raumes, Proc., **21** (1945), 171 ; Sur la déformation infinitésimale des sous-espaces dans un espace affine, ibidem, 248 ; Sur la déformation infinitésimale tangentielle d'un sous-espace, ibidem, 261; Quelques remarques sur un article de M.N. Coburn intitulé " A characterization of Schouten's and Hayden's deformation methods," ibidem, 330.

2) M. S. Knebelman: Collineations and motions in generalized spaces, Amer. Journ. of Math., **51** (1929), 527–564.

3) S. Hokari: Winkeltreue Transformationen und Bewegungen im Finslerschen Raume, Journal of the Faculty of Science, Hokkaido Imp. Univ., **5** (1936), 1–8.

4) E. T. Davies: Lie derivation in generalized metric space, Annali di Mat., **18** (1939), 261–274.

5) J. Douglas: The general geometry of paths, Annals of Math., **29** (1928), 143–168.

$(x^1, x^2, \dots, x^n)$, the paths are represented as the integral curves of a system of differential equations

$$(1.1) \qquad \frac{d\dot{x}^\lambda}{ds} + H^\lambda(x, \dot{x}) = 0 \,, \quad \dot{x}^\lambda = \frac{dx^\lambda}{ds} \,,$$

where $H^\lambda(x, \dot{x})$ are homogeneous functions of the second degree in $\dot{x}^\lambda$.

Thus the functions $\Gamma^\lambda_{\mu\nu}(x, \dot{x})$ defined by

$$(1.2) \qquad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} \frac{\partial^2 H^\lambda}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$$

are symmetric with respect to the indices μ and ν, and are homogeneous of degree zero in $\dot{x}^\lambda$.

The law of transformation of the functions $\Gamma^\lambda_{\mu\nu}$ under a coordinate transformation $x^\lambda = x^\lambda(x^\alpha)$ may be obtained from (1.1) assuming that the parameter s is invariant:

$$(1.3) \qquad \overline{\Gamma}^\alpha_{\beta\gamma} = \frac{\partial \overline{x}^\alpha}{\partial x^\lambda} \left(\frac{\partial x^\mu}{\partial \overline{x}^\beta} \frac{\partial x^\nu}{\partial \overline{x}^\gamma} \Gamma^\lambda_{\mu\nu} + \frac{\partial^2 x^\lambda}{\partial \overline{x}^\beta \partial \overline{x}^\gamma} \right),$$

from which, differentiating with respect to $\dot{\overline{x}}^\delta$, and remarking that

$$\dot{x}^\omega = \frac{\partial x^\omega}{\partial \overline{x}^\delta} \dot{\overline{x}}^\delta \,,$$

we have

$$(1.4) \qquad \overline{\Gamma}^\alpha_{\beta\gamma/\delta} = \frac{\partial \overline{x}^\alpha}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \overline{x}^\beta} \frac{\partial x^\nu}{\partial \overline{x}^\gamma} \frac{\partial x^\omega}{\partial \overline{x}^\delta} \Gamma^\lambda_{\mu\nu/\omega} \,,$$

where the solidus indicates the partial differentiation with respect to $\dot{x}^\omega$.

The transformation law of $\dot{x}^\lambda$ being given by $\dot{\overline{x}}^\lambda = \frac{\partial \overline{x}^\lambda}{\partial x^\alpha} \dot{x}^\alpha$, it will be easily seen that the partial differentiation $T^\lambda_{\cdot\mu\nu/\omega}$ of an arbitrary tensor $T^\lambda_{\cdot\mu\nu}$ with respect to $\dot{x}^\omega$ gives a new tensor having one more covariant index.

The equations (1.4) show us that the quantities defined by

$$(1.5) \qquad \Gamma^\lambda_{\mu\nu\omega} = \Gamma^\lambda_{\mu\nu/\omega}$$

are components of a mixed tensor symmetric with respect to three lower indices.

The law of transformation of the functions $\Gamma^\lambda_{\mu\nu}$ being given by (1.3), we can define the covariant derivative of any tensor $T^\lambda_{\cdot\mu\nu}$ by

$$(1.6) \quad T^\lambda_{\cdot\mu\nu;\omega} = T^\lambda_{\cdot\mu\nu,\omega} - T^\lambda_{\cdot\mu\nu/\alpha} \Gamma^\alpha_{\beta\omega} \dot{x}^\beta + T^\alpha_{\cdot\mu\nu} \Gamma^\lambda_{\alpha\omega} - T^\lambda_{\cdot\alpha\nu} \Gamma^\alpha_{\mu\omega} - T^\lambda_{\cdot\mu\alpha} \Gamma^\alpha_{\nu\omega} \,,$$

thus, the semicolon indicating the covariant derivative while the comma the ordinary derivative with respect to x^ω. The parallel propagation of a vector being defined by the vanishing of its covariant derivative, the functions $\Gamma^\lambda_{\mu\nu}$ are called the components of the affine connection in the general space of paths.

Applying the definition of covariant differentiation to the $\dot{x}^\lambda$, we find that

$$(\dot{x}^\lambda)_{;\nu} = 0 \,,$$

say, $\dot{x}^\lambda$ is a parallel vector field with respect to this affine connection.

The operation of covariant differentiation is not commutative. The formulae of Ricci

$$(1.7)\qquad T^{\lambda}_{\bullet\mu\nu;\omega;\sigma}-T^{\lambda}_{\bullet\mu\nu;\sigma;\omega}$$
$$=T^{\alpha}_{\bullet\mu\nu}R^{\lambda}_{\bullet\alpha\omega\sigma}-T^{\lambda}_{\bullet\alpha\nu}R^{\alpha}_{\bullet\mu\omega\sigma}-T^{\lambda}_{\bullet\mu\alpha}R^{\alpha}_{\bullet\nu\omega\sigma}-T^{\lambda}_{\bullet\mu\nu/\alpha}R^{\alpha}_{\bullet\beta\omega\sigma}\dot{x}^{\beta}$$

may be obtained by a straightforward calculation, where

$$(1.8)\qquad R^{\lambda}_{\bullet\mu\nu\omega}=(\Gamma^{\lambda}_{\mu\nu,\omega}-\Gamma^{\lambda}_{\mu\nu/\alpha}\Gamma^{\alpha}_{\beta\omega}\dot{x}^{\beta})-(\Gamma^{\lambda}_{\mu\omega,\nu}-\Gamma^{\lambda}_{\mu\omega/\alpha}\Gamma^{\alpha}_{\beta\nu}\dot{x}^{\beta})$$
$$+\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\omega}-\Gamma^{\alpha}_{\mu\omega}\Gamma^{\lambda}_{\alpha\nu}$$

are the components of the curvature tensor of the space, and satisfy the following identities

$$(1.9)\qquad R^{\lambda}_{\bullet\mu\nu\omega}+R^{\lambda}_{\bullet\mu\omega\nu}=0\,,$$

$$(1.10)\qquad R^{\lambda}_{\bullet\mu\nu\omega}+R^{\lambda}_{\bullet\nu\omega\mu}+R^{\lambda}_{\bullet\omega\mu\nu}=0\,,$$

$$(1.11)\qquad R^{\lambda}_{\bullet\mu\nu\omega;\sigma}+R^{\lambda}_{\bullet\mu\omega\sigma;\nu}+R^{\lambda}_{\bullet\mu\sigma\nu;\omega}$$
$$+(\Gamma^{\lambda}_{\mu\nu\alpha}R^{\alpha}_{\bullet\beta\omega\sigma}+\Gamma^{\lambda}_{\mu\omega\alpha}R^{\alpha}_{\bullet\beta\sigma\nu}+\Gamma^{\lambda}_{\mu\sigma\alpha}R^{\alpha}_{\bullet\beta\nu\omega})\dot{x}^{\beta}=0\,.$$

The last two are identities of Bianchi.

If we commute the operations of covariant differentiation and of partial differentiation with respect to $\dot{x}^{\sigma}$, we find

$$(1.12)\qquad T^{\lambda}_{\bullet\mu\nu;\omega/\sigma}-T^{\lambda}_{\bullet\mu\nu/\sigma;\omega}=T^{\alpha}_{\bullet\mu\nu}\Gamma^{\lambda}_{\alpha\omega\sigma}-T^{\lambda}_{\bullet\alpha\nu}\Gamma^{\alpha}_{\mu\omega\sigma}-T^{\lambda}_{\bullet\mu\alpha}\Gamma^{\alpha}_{\nu\omega\sigma}\,.$$

Concerning these operators, we have the identities:

$$(1.13)\qquad \Gamma^{\lambda}_{\mu\nu\omega;\sigma}-\Gamma^{\lambda}_{\mu\nu\sigma;\omega}=R^{\lambda}_{\bullet\mu\omega\sigma/\nu}\,,$$

$$(1.14)\qquad R^{\lambda}_{\bullet\mu\nu\omega/\sigma}+R^{\lambda}_{\bullet\mu\omega\sigma/\nu}+R^{\lambda}_{\bullet\mu\sigma\nu/\omega}=0\,.$$

§ 2. *The Lie derivatives of tensors.* Let us consider an infinitesimal deformation of the space

$$(2.1)\qquad \bar{x}^{\lambda}=x^{\lambda}+\xi^{\lambda}(x)dt\,,$$

which displaces the point x^{λ} to the infinitely near point $x^{\lambda}+\xi^{\lambda}(x)dt$, where $\xi^{\lambda}(x)$ is a vector field defined at every point of the space and dt is an infinitesimal quantity. We shall concerned with only the quantities of the first order with respect to dt.

If we consider a scalar field $f(x,\dot{x})$ at the point x^{λ} and displace it from the point x^{λ} to an infinitely near point $\bar{x}^{\lambda}$, we have a value $\bar{f}(\bar{x},\dot{\bar{x}})$ of $f(x,\dot{x})$ at $\bar{x}^{\lambda}$. But the $f(x,\dot{x})$ being a scalar, we may assume that we have

$$\bar{f}(\bar{x},\dot{\bar{x}})=f(x,\dot{x})\,.$$

Thus putting

$$(2.2)\qquad Df=f(\bar{x},\dot{\bar{x}})-\bar{f}(\bar{x},\dot{\bar{x}})\,,$$

we have

$$(2.3)\qquad Df=df=[f_{,\nu}\xi^{\nu}+f_{/\alpha}\xi^{\alpha}{}_{,\nu}\dot{x}^{\nu}]dt,$$

or, in a tensor form,

$$(2.4)\qquad Df=[f_{;\nu}\xi^{\nu}+f_{/\alpha}\xi^{\alpha}{}_{;\nu}\dot{x}^{\nu}]dt\,.$$

We shall call Df the Lie derivative of the scalar $f(x, \dot{x})$.

The Lie derivative of a scalar field being thus defined, we shall consider next a contravariant vector field $v^\lambda(x, \dot{x})$ and define its Lie derivative in the following manner.

We consider a point $P(x^\lambda)$ and a point $Q(x^\lambda + v^\lambda\varepsilon)$ which is infinitely near to $P(x^\lambda)$ and lies on the direction given by v^λ, then we displace these two points by the infinitesimal deformation (2.1):

$$P(x^\lambda) \to P'(x^\lambda + \xi^\lambda dt),$$
$$Q(x^\lambda + v^\lambda\varepsilon) \to Q'(x^\lambda + v^\lambda\varepsilon + \xi^\lambda dt + \xi^\lambda{}_{,\nu} v^\nu \varepsilon dt),$$

thus, we have a vector $\bar{v}^\lambda(\bar{x}, \bar{\dot{x}}) = \overrightarrow{P'Q'}/\varepsilon$ at $\bar{x}^\lambda$ whose components are given by

$$\bar{v}^\lambda(\bar{x}, \bar{\dot{x}}) = v^\lambda + \xi^\lambda{}_{,\nu} v^\nu dt.$$

On the other hand, we have, at Q,

$$v^\lambda(\bar{x}, \bar{\dot{x}}) = v^\lambda + [v^\lambda{}_{,\nu}\xi^\nu + v^\lambda{}_{/\alpha}\xi^\alpha{}_{,\nu}\dot{x}^\nu]dt,$$

thus, putting

$$Dv^\lambda = v^\lambda(\bar{x}, \bar{\dot{x}}) - \bar{v}^\lambda(\bar{x}, \bar{\dot{x}}), \tag{2.5}$$

we have

$$Dv^\lambda = [v^\lambda{}_{,\nu}\xi^\nu + v^\lambda{}_{/\alpha}\xi^\alpha{}_{,\nu}\dot{x}^\nu - \xi^\lambda{}_{,\nu} v^\nu]dt, \tag{2.6}$$

or, in tensor form,

$$Dv^\lambda = [v^\lambda{}_{;\nu}\xi^\nu + v^\lambda{}_{/\alpha}\xi^\alpha{}_{;\nu}\dot{x}^\nu - \xi^\lambda{}_{;\nu} v^\nu]dt. \tag{2.7}$$

This gives the Lie derivative of a contravariant vector field v^λ.

Following this definition of Lie derivative, the Lie derivative $D\dot{x}^\lambda$ of the $\dot{x}^\nu$ is identically zero.

To obtain the Lie derivative of a covariant vector field $u_\mu(x, \dot{x})$, we take an arbitrary contravariant vector field $v^\lambda(x, \dot{x})$ and form a scalar $u_\lambda v^\lambda$. Then from the assumption for a scalar field, we have

$$\bar{u}_\lambda(\bar{x}, \bar{\dot{x}})\bar{v}^\lambda(\bar{x}, \bar{\dot{x}}) = u_\lambda(x, \dot{x})v^\lambda(x, \dot{x}),$$

form which we find

$$D(u_\lambda v^\lambda) = (Du_\lambda)v^\lambda + u_\lambda(Dv^\lambda). \tag{2.8}$$

Substituting (2.3) and (2.6) in this equation, we find

$$[(u_\lambda v^\lambda)_{,\nu}\xi^\nu + (u_\nu v^\lambda)_{/\alpha}\xi^\alpha{}_{,\nu}\dot{x}^\nu]dt$$
$$= (Du_\lambda)v^\lambda + u_\lambda[v^\lambda{}_{,\nu}\xi^\nu + v^\lambda{}_{/\alpha}\xi^\alpha{}_{,\nu}\dot{x}^\nu - \xi^\lambda{}_{,\nu} v^\nu]dt,$$

from which we have

$$Du_\lambda = [u_{\lambda,\nu}\xi^\nu + u_{\lambda/\alpha}\xi^\alpha{}_{,\nu}\dot{x}^\nu + \xi^\alpha{}_{,\lambda}u_\alpha]dt, \tag{2.9}$$

or, in tensor form

$$Du_\lambda = [u_{\lambda;\nu}\xi^\nu + u_{\lambda/\alpha}\xi^\alpha{}_{;\nu}\dot{x}^\nu + \xi^\alpha{}_{;\lambda}u_\alpha]dt, \tag{2.10}$$

the vector field v^λ being quite arbitrary.

The tensor form (2.10) may also be obtained by substitution of (2.4) and (2.7) in (2.8).

The Lie derivatives of a scalar, a contravariant vector and a covariant vector being thus obtained, the Lie derivative of an arbitrary tensor field of any type, say, $T^{\lambda}_{\cdot\mu\nu}$ may be obtained from the equation

$$D(T^{\lambda}_{\cdot\mu\nu}u_{\lambda}v^{\mu}w^{\nu})=(DT^{\lambda}_{\cdot\mu\nu})u_{\lambda}v^{\mu}w^{\nu}+T^{\lambda}_{\cdot\mu\nu}(Du_{\lambda})v^{\mu}w^{\nu}+T^{\lambda}_{\cdot\mu\nu}u_{\lambda}(Dv^{\mu})w^{\nu}$$
$$+T^{\lambda}_{\cdot\mu\nu}u_{\lambda}v^{\mu}(Dw^{\nu}),$$

where u_{λ}, v^{λ} and w^{λ} are arbitrary vectors. The Lie derivative $DT^{\lambda}_{\cdot\mu\nu}$ of $T^{\lambda}_{\cdot\mu\nu}$ thus obtained is

$$(2.11)\quad DT^{\lambda}_{\cdot\mu\nu}=[T^{\lambda}_{\cdot\mu\nu,\alpha}\xi^{\alpha}+T^{\lambda}_{\cdot\mu\nu/\alpha}\xi^{\alpha}_{,\omega}\dot{x}^{\omega}-T^{\alpha}_{\cdot\mu\nu}\xi^{\lambda}_{,\alpha}+T^{\lambda}_{\cdot\alpha\nu}\xi^{\alpha}_{,\mu}+T^{\lambda}_{\cdot\mu\alpha}\xi^{\alpha}_{,\nu}]dt,$$

or, in tensor form

$$(2.12)\quad DT^{\lambda}_{\cdot\mu\nu}=[T^{\lambda}_{\cdot\mu\nu;\alpha}\xi^{\alpha}+T^{\lambda}_{\cdot\mu\nu/\alpha}\xi^{\alpha}_{;\omega}\dot{x}^{\omega}-T^{\alpha}_{\cdot\mu\nu}\xi^{\lambda}_{;\alpha}+T^{\lambda}_{\cdot\alpha\nu}\xi^{\alpha}_{;\mu}+T^{\lambda}_{\cdot\mu\alpha}\xi^{\alpha}_{;\nu}]dt.$$

§ 3. *The Lie derivative of the affine connection* $\Gamma^{\lambda}_{\mu\nu}$. To obtain the Lie derivative of the affine connection $\Gamma^{\lambda}_{\mu\nu}$, we consider first a contravariant vector $v^{\lambda}(x,\dot{x})$ and displace it parallelly from the point x^{λ} to a nearby point $x^{\lambda}+dx^{\lambda}$, then, we obtain at $x^{\lambda}+dx^{\lambda}$,

$$(3.1)\qquad \bar{v}^{\lambda}(x+dx,\dot{x}+d\dot{x})$$
$$=v^{\lambda}+v^{\lambda}_{/\alpha}\Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}dx^{\nu}-\Gamma^{\lambda}_{\mu\nu}v^{\mu}dx^{\nu}.$$

We displace next the vectors $v^{\lambda}(x,\dot{x})$ and $\bar{v}^{\lambda}(x+dx,\dot{x}+d\dot{x})$ by the infinitesimal deformation (2.1). Then we have two vectors

$$\bar{v}^{\lambda}(\bar{x},\bar{\dot{x}})=v^{\lambda}+\xi^{\lambda}_{,\nu}v^{\nu}dt$$

at $\bar{x}^{\lambda}=x^{\lambda}+\xi^{\lambda}dt$ and

$$(3.2)\qquad \bar{\bar{v}}^{\lambda}(\bar{x}+d\bar{x},\bar{\dot{x}}+d\bar{\dot{x}})=v^{\lambda}+v^{\lambda}_{/\alpha}\Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}dx^{\nu}-\Gamma^{\lambda}_{\mu\nu}v^{\mu}dx^{\nu}$$
$$+(\xi^{\lambda}_{,\mu}+\xi^{\lambda}_{,\mu,\nu}dx^{\nu})(v^{\mu}+v^{\mu}_{/\alpha}\Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\beta}dx^{\gamma}-\Gamma^{\mu}_{\beta\gamma}v^{\beta}dx^{\gamma})dt$$
$$=v^{\lambda}+v^{\lambda}_{/\alpha}\Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}dx^{\nu}-\Gamma^{\lambda}_{\mu\nu}v^{\mu}dx^{\nu}$$
$$+[\xi^{\lambda}_{;\alpha}v^{\alpha}+\xi^{\lambda}_{,\mu}v^{\mu}_{/\alpha}\Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\beta}dx^{\gamma}-\xi^{\lambda}_{,\mu}\Gamma^{\mu}_{\beta\gamma}v^{\beta}dx^{\gamma}+\xi^{\lambda}_{,\mu,\nu}v^{\mu}dx^{\nu}]dt$$

at $\bar{x}^{\lambda}+d\bar{x}^{\lambda}$ respectively.

Now, to obtain the displaced values $\bar{\Gamma}^{\lambda}_{\mu\nu}(\bar{x},\bar{\dot{x}})$ of the affine connection $\Gamma^{\lambda}_{\mu\nu}(x,\dot{x})$, we assume that the vector $\bar{v}^{\lambda}(\bar{x},\bar{\dot{x}})$ is parallel to the vector $\bar{\bar{v}}^{\lambda}(\bar{x}+d\bar{x},\bar{\dot{x}}+d\bar{\dot{x}})$ with respect to the varied affine connection $\bar{\Gamma}^{\lambda}_{\mu\nu}(\bar{x},\bar{\dot{x}})$. Thus, we obtain

$$(3.3)\qquad \bar{\bar{v}}^{\lambda}(\bar{x}+d\bar{x},\bar{\dot{x}}+d\bar{\dot{x}})=\bar{v}^{\lambda}(\bar{x},\bar{\dot{x}})+\bar{v}^{\lambda}(\bar{x},\bar{\dot{x}})_{/\alpha}\bar{\Gamma}^{\alpha}_{\mu\nu}(\bar{x},\bar{\dot{x}})\bar{\dot{x}}^{\mu}d\bar{x}^{\nu}$$
$$-\bar{\Gamma}^{\lambda}_{\mu\nu}(\bar{x},\bar{\dot{x}})\bar{v}^{\mu}(\bar{x},\bar{\dot{x}})d\bar{x}^{\nu}.$$

Put

$$(3.4)\qquad D\Gamma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}(\bar{x},\bar{\dot{x}})-\bar{\Gamma}^{\lambda}_{\mu\nu}(\bar{x},\bar{\dot{x}}),$$

and subtitute (3.1), (3.2) and

$$\bar{\Gamma}^{\lambda}_{\mu\nu}(\bar{x},\bar{\dot{x}})=\Gamma^{\lambda}_{\mu\nu}+\Gamma^{\lambda}_{\mu\nu,\omega}\xi^{\omega}dt+\Gamma^{\lambda}_{\mu\nu/\alpha}\xi^{\alpha}_{,\omega}\dot{x}^{\omega}dt-D\Gamma^{\lambda}_{\mu\nu}$$

in (3.3), we find

$$v^\lambda+v^\lambda{}_{/\alpha}\Gamma^\alpha_{\mu\nu}\dot{x}^\mu dx^\nu-\Gamma^\lambda_{\mu\nu}v^\mu dx^\nu+[\xi^\lambda{}_{,\alpha}v^\alpha+\xi^\lambda{}_{,\mu}v^\mu{}_{/\alpha}\Gamma^\alpha_{\beta\gamma}\dot{x}^\beta dx^\gamma-\xi^\lambda{}_{,\mu}\Gamma^\mu_{\beta\gamma}v^\beta dx^\gamma$$
$$+\xi^\lambda{}_{,\mu,\nu}v^\mu dx^\nu]dt$$
$$=v^\lambda+\xi^\lambda{}_{,\nu}v^\nu dt+(v^\lambda{}_{/\alpha}+\xi^\lambda{}_{,\gamma}v^\gamma{}_{/\alpha}dt)(\Gamma^\alpha_{\mu\nu}+\Gamma^\alpha_{\mu\nu,\omega}\xi^\omega dt+\Gamma^\alpha_{\mu\nu/\beta}\xi^\beta{}_{,\omega}\dot{x}^\omega dt)$$
$$\times(\dot{x}^\mu+\xi^\mu{}_{,\gamma}\dot{x}^\gamma dt)(dx^\nu+\xi^\nu{}_{,\delta}dx^\delta)$$
$$-(\Gamma^\lambda_{\mu\nu}+\Gamma^\lambda_{\mu\nu,\omega}\xi^\omega dt+\Gamma^\lambda_{\mu\nu/\alpha}\xi^\alpha{}_{,\omega}\dot{x}^\omega dt-D\Gamma^\lambda_{\mu\nu})(v^\mu+\xi^\mu{}_{,\gamma}v^\gamma dt)(dx^\nu+\xi^\nu{}_{,\delta}dx^\delta)\,,$$

from which we have

$$D\Gamma^\lambda_{\mu\nu}=[\xi^\lambda{}_{,\mu,\nu}-\xi^\lambda{}_{,\alpha}\Gamma^\alpha_{\mu\nu}+\xi^\alpha{}_{,\mu}\Gamma^\lambda_{\alpha\nu}+\xi^\alpha{}_{,\nu}\Gamma^\lambda_{\alpha\mu}+\Gamma^\lambda_{\mu\nu,\omega}\xi^\omega+\Gamma^\lambda_{\mu\nu/\alpha}\xi^\alpha{}_{,\omega}\dot{x}^\omega]dt\,, \tag{3.5}$$

or, in tensor form

$$D\Gamma^\lambda_{\mu\nu}=[\xi^\lambda{}_{;\mu;\nu}+R^\lambda{}_{\cdot\mu\nu\omega}\xi^\omega+\Gamma^\lambda_{\mu\nu\omega}\xi^\omega{}_{;\alpha}\dot{x}^\alpha]dt\,. \tag{3.6}$$

The equations (3.5) and (3.6) give the Lie derivative of the affine connection $\Gamma^\lambda_{\mu\nu}$.

The assumption on the deformed affine connection $\Gamma^\lambda_{\mu\nu}$ may also be expressed by the formulae

$$v^\lambda+\delta v^\lambda+D(v^\lambda+\delta v^\lambda)=v^\lambda+Dv^\lambda+\bar{\delta}(v^\lambda+Dv^\lambda)\,, \tag{3.7}$$

where δ and $\bar{\delta}$ denote the covariant differential with respect to the original affine connection and the deformed affine connection respectively.

Thus from (3.7), we obtain

$$D\delta v^\lambda-\delta Dv^\lambda=v^\mu(D\Gamma^\lambda_{\mu\nu})dx^\nu-v^\lambda{}_{/\alpha}(D\Gamma^\alpha_{\mu\nu})\dot{x}^\mu dx^\nu\,. \tag{3.8}$$

The Lie derivative of the dx^ν being zero, we have, from (3.8),

$$D(v^\lambda{}_{;\nu})-(Dv^\lambda)_{;\nu}=v^\mu(D\Gamma^\lambda_{\mu\nu})-v^\lambda{}_{/\alpha}(D\Gamma^\alpha_{\mu\nu})\dot{x}^\mu\,. \tag{3.9}$$

The analoguous formulae for a covariant vector and a general tensor are respectively

$$D(u_{\mu;\nu})-(Du_\mu)_{;\nu}=-u_\lambda(D\Gamma^\lambda_{\mu\nu})-u_{\mu/\alpha}(D\Gamma^\alpha_{\beta\nu})\dot{x}^\beta \tag{3.10}$$

and

$$D(T^\lambda{}_{\cdot\mu\nu;\omega})-(DT^\lambda{}_{\cdot\mu\nu})_{;\omega} \tag{3.11}$$
$$=T^\alpha{}_{\cdot\mu\nu}(D\Gamma^\lambda_{\alpha\omega})-T^\lambda{}_{\cdot\alpha\nu}(D\Gamma^\alpha_{\mu\omega})-T^\lambda{}_{\cdot\mu\alpha}(D\Gamma^\alpha_{\nu\omega})-T^\lambda{}_{\cdot\mu\nu/\alpha}(D\Gamma^\alpha_{\beta\omega})\dot{x}^\beta\,.$$

These formulae give the law of commutation of the two operations, the Lie differentiation and covariant differentiation.

If we replace the covariant differentiation by the partial differentiation with respect to $\dot{x}^\nu$, we obtain

$$D(v^\lambda{}_{/\nu})-(Dv^\lambda)_{/\nu}=0\,, \tag{3.12}$$

$$D(u_{\mu/\nu})-(Du_\mu)_{/\nu}=0\,, \tag{3.13}$$

$$D(T^\lambda{}_{\cdot\mu\nu/\omega})-(DT^\lambda{}_{\cdot\mu\nu})_{/\omega}=0\,. \tag{3.14}$$

§ 4. *Collineations in the generalized space.* Consider a path, that is, a curve $x^\lambda(s)$ satisfying the differential equations

$$\frac{d^2x^\lambda}{ds^2}+\Gamma^\lambda_{\mu\nu}\left(x,\frac{dx}{ds}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0\,. \tag{4.1}$$

If we displace the every point of the curve by an infinitesimal deformation

$$(4.2) \qquad \bar{x}^\lambda(s) = x^\lambda(s) + \xi^\lambda(s)dt,$$

we have a new curve $\bar{x}^\lambda(s)$. We shall seek for the necessary and sufficient condition that the new curve thus obtained be also a path of the space. In order that it may be the case, we must have the differential equations of the form

$$\frac{d^2\bar{x}^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\left(\bar{x}, \frac{d\bar{x}}{ds}\right)\frac{d\bar{x}^\mu}{ds}\frac{d\bar{x}^\nu}{ds} = \rho\left(\bar{x}, \frac{d\bar{x}}{ds}\right)dt\frac{d\bar{x}^\lambda}{ds}.$$

Substituting (4.2) in these equations and taking account of (4.1), we find

$$(4.3) \qquad \frac{\delta^2\xi^\lambda}{ds^2} + R^\lambda_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega + \Gamma^\lambda_{\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\delta\xi^\omega}{ds} = \rho\frac{dx^\lambda}{ds},$$

which define the geodesic deviation in the general space of paths.

Now suppose that the ξ^λ is a contravariant vector field defined in the space. Then the equations (4.3) may be written as

$$(4.4) \qquad \left[\xi^\lambda{}_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega + \Gamma^\lambda_{\mu\nu\alpha}\xi^\alpha{}_{;\omega}\frac{dx^\omega}{ds}\right]\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \rho\frac{dx^\lambda}{ds}.$$

If every path in the space is transformed into another path by the infinitesimal deformation

$$\bar{x}^\lambda = x^\lambda + \xi^\lambda(x^\alpha)dt,$$

we shall call such an infinitesimal deformation the infinitesimal projective collineation of the space.

In order that the space admit an infinitesimal projective collineation, the equations (4.4) must be satisfied for every value of $\dfrac{dx^\lambda}{ds}$, thus we have[6]

$$(4.5) \qquad \xi^\lambda{}_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega + \Gamma^\lambda_{\mu\nu\alpha}\xi^\alpha{}_{;\omega}\dot{x}^\omega = \rho_{/\mu}\delta^\lambda_\nu + \rho_{/\nu}\delta^\lambda_\mu + \rho_{/\mu/\nu}\dot{x}^\lambda,$$

this condition is also sufficient.

If we suppose that the parameter s which is affine for the original path is also affine for the displaced path, we have $\rho=0$, and we find, instead of (4.5),

$$(4.6) \qquad \xi^\lambda{}_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega + \Gamma^\lambda_{\mu\nu\alpha}\xi^\alpha{}_{;\omega}\dot{x}^\omega = 0.$$

In this case, we say that the space admits an infinitesimal affine collineation.

§ 5. *The deformed space.* We have defined the Lie derivative Dv^λ of a contravariant vector field v^λ by the formulae

$$Dv^\lambda = v^\lambda(\bar{x}, \dot{\bar{x}}) - \bar{v}^\lambda(\bar{x}, \dot{\bar{x}}).$$

But, strictly speaking, the Dv^λ is defined at $\bar{x}^\lambda$ and not at x^λ.

To find the value of Dv^λ at x^λ, we may replace $\bar{x}^\lambda$ by x^λ and dt by $-dt$ in the above equation, then we have

$$-Dv^\lambda = v^\lambda(x, \dot{x}) - \bar{v}^\lambda(x, \dot{x})$$

or

6) M. S. Knebelman, loc. cit., p. 541.

$$\bar{v}^\lambda(x, \dot{x}) = v^\lambda(x, \dot{x}) + Dv^\lambda. \tag{5.1}$$

Similarly we have

$$\bar{u}_\mu(x, \dot{x}) = u_\mu(x, \dot{x}) + Du_\mu \tag{5.2}$$

for a covariant vector field,

$$\bar{T}^\lambda_{\cdot\mu\nu}(x, \dot{x}) = T^\lambda_{\cdot\mu\nu}(x, \dot{x}) + DT^\lambda_{\cdot\mu\nu} \tag{5.3}$$

for a mixed tensor $T^\lambda_{\cdot\mu\nu}$ and

$$\bar{\Gamma}^\lambda_{\mu\nu}(x, \dot{x}) = \Gamma^\lambda_{\mu\nu}(x, \dot{x}) + D\Gamma^\lambda_{\mu\nu} \tag{5.4}$$

for the affine connection.

We shall call the deformed space, the space whose parameters of connection are given by $\bar{\Gamma}^\lambda_{\mu\nu}$.

Denoting by $\bar{\delta}$ the covariant derivative with respect to $\bar{\Gamma}^\lambda_{\mu\nu}$, we have, from the definition of $\bar{\Gamma}^\lambda_{\mu\nu}$,

$$\bar{v}^\lambda + \bar{\delta}\bar{v}^\lambda = v^\lambda + \delta v^\lambda + D(v^\lambda + \delta v^\lambda),$$

from which

$$\bar{\delta}\bar{v}^\lambda = \delta v^\lambda + D\delta v^\lambda. \tag{5.5}$$

Similarly we have

$$\bar{\delta}\bar{u}_\mu = \delta u_\mu + D\delta u_\mu \tag{5.6}$$

for a covariant vector field v_μ and

$$\bar{\delta}\bar{T}^\lambda_{\cdot\mu\nu} = \delta T_{\cdot\mu\nu} + D\delta T_{\cdot\mu\nu} \tag{5.7}$$

for a mixed tensor field $T^\lambda_{\cdot\mu\nu}$.

These formulae may be written as

$$\bar{v}^\lambda{}_{;\nu} = v^\lambda{}_{;\nu} + D(v^\lambda{}_{;\nu}), \tag{5.8}$$

$$\bar{u}_{\mu;\nu} = u_{\mu;\nu} + D(u_{\mu;\nu}), \tag{5.9}$$

$$\bar{T}^\lambda_{\cdot\mu\nu\omega} = T^\lambda_{\cdot\mu\nu;;\omega} + D(T^\lambda_{\cdot\mu\nu;\omega}) \tag{5.10}$$

respectively.

Now, the formulae of Ricci are

$$v^\lambda{}_{;\nu;\omega} - v^\lambda{}_{;\omega;\nu} = v^\mu R^\lambda_{\cdot\mu\nu\omega} - v^\lambda{}_{/\alpha} R^\alpha_{\cdot\beta\nu\omega}\dot{x}^\beta$$

for the original space and

$$\bar{v}^\lambda{}_{;\nu;\omega} - \bar{v}^\lambda{}_{;\omega;\nu} = \bar{v}^\mu \bar{R}^\lambda_{\cdot\mu\nu\omega} - \bar{v}^\lambda{}_{/\alpha} \bar{R}^\alpha_{\cdot\beta\nu\omega}\dot{\bar{x}}^\beta$$

for the deformed space.

On the other hand, we have

$$\bar{v}^\lambda{}_{;\nu;\omega} = v^\lambda{}_{;\nu;\omega} + D(v^\lambda{}_{;\nu;\omega}),$$

consequently

$$\bar{v}^\mu \bar{R}^\lambda_{\cdot\mu\nu\omega} - \bar{v}^\lambda{}_{/\alpha}\bar{R}^\alpha_{\cdot\beta\nu\omega}\dot{\bar{x}}^\beta = v^\mu R^\lambda_{\cdot\mu\nu\omega} - v^\lambda{}_{/\alpha} R^\alpha_{\cdot\beta\nu\omega}\dot{x}^\beta + D(v^\mu R^\lambda_{\cdot\mu\nu\omega} - v^\lambda{}_{/\alpha}R^\alpha_{\cdot\beta\nu\omega}\dot{x}^\beta).$$

Remembering that $\bar{v}^\lambda = v^\lambda + Dv^\lambda$ and $\bar{v}^\lambda{}_{/\alpha} = v^\lambda{}_{/\alpha} + Dv^\lambda{}_{/\alpha}$, we have

$$\bar{R}^\lambda_{\cdot\mu\nu\omega} = R^\lambda_{\cdot\mu\nu\omega} + DR^\lambda_{\mu\nu\omega}. \tag{5.11}$$

The formula (5.11) may also be obtained in the following manner.

Substituting $\bar{\Gamma}^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}+D\Gamma^{\lambda}_{\mu\nu}$ in the equation

$$\bar{R}^{\lambda}_{\cdot\mu\nu\omega}=(\bar{\Gamma}^{\lambda}_{\mu\nu,\omega}-\bar{\Gamma}^{\lambda}_{\mu\nu/\alpha}\bar{\Gamma}^{\alpha}_{\beta\omega}\dot{\bar{x}}^{\beta})-(\bar{\Gamma}^{\lambda}_{\mu\omega,\nu}-\bar{\Gamma}^{\lambda}_{\mu\omega/\alpha}\bar{\Gamma}^{\alpha}_{\beta\nu}\dot{\bar{x}}^{\beta})+\bar{\Gamma}^{\alpha}_{\mu\nu}\bar{\Gamma}^{\lambda}_{\alpha\omega}-\bar{\Gamma}^{\alpha}_{\mu\omega}\bar{\Gamma}^{\lambda}_{\alpha\nu},$$

we obtain

$$(5.12)\quad \bar{R}^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\mu\nu\omega}+(D\Gamma^{\lambda}_{\mu\nu})_{;\omega}-(D\Gamma^{\lambda}_{\mu\omega})_{;\nu}-\Gamma^{\lambda}_{\mu\nu\alpha}(D\Gamma^{\alpha}_{\beta\omega})\dot{x}^{\beta}+\Gamma^{\lambda}_{\mu\omega\alpha}(D\Gamma^{\alpha}_{\beta\nu})\dot{x}^{\beta}.$$

Substituting (3.6) in this equation, we find

$$\begin{aligned}\bar{R}^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\mu\nu\omega}&+[\xi^{\lambda}{}_{;\mu;\nu;\omega}+R^{\lambda}_{\cdot\mu\nu\alpha;\omega}\xi^{\alpha}+R^{\lambda}_{\cdot\mu\nu\alpha}\xi^{\alpha}{}_{;\omega}+\Gamma^{\lambda}_{\mu\nu\alpha;\omega}\xi^{\lambda}{}_{;\beta}\dot{x}^{\beta}\\&\qquad+\Gamma^{\lambda}_{\mu\nu\alpha}\xi^{\alpha}{}_{;\beta;\omega}\dot{x}^{\beta}]dt\\&-[\xi^{\lambda}{}_{;\mu;\omega;\nu}+R^{\lambda}_{\cdot\mu\omega\alpha;\nu}\xi^{\alpha}+R^{\lambda}_{\cdot\mu\omega\alpha}\xi^{\alpha}{}_{;\nu}+\Gamma^{\lambda}_{\mu\omega\alpha;\nu}\xi^{\alpha}{}_{;\beta}\dot{x}^{\beta}+\Gamma^{\lambda}_{\mu\omega\alpha}\xi^{\alpha}{}_{;\beta;\nu}\dot{x}^{\beta}]dt\\&-\Gamma^{\lambda}_{\mu\nu\alpha}(\xi^{\alpha}{}_{;\beta;\omega}+R^{\alpha}_{\cdot\beta\omega\tau}\xi^{\tau}+\Gamma^{\alpha}_{\beta\omega\tau}\xi^{\tau}{}_{;\delta}\dot{x}^{\delta}]dt\\&+\Gamma^{\lambda}_{\mu\omega\alpha}(\xi^{\alpha}{}_{;\beta;\nu}+R^{\lambda}_{\cdot\beta\nu\tau}\xi^{\tau}+\Gamma^{\alpha}_{\beta\nu\tau}\xi^{\tau}{}_{;\delta}\dot{x}^{\delta}]dt,\end{aligned}$$

from which

$$(5.13)\quad \begin{aligned}\bar{R}^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\mu\nu\omega}&+[R^{\lambda}_{\cdot\mu\nu\omega;\alpha}\xi^{\alpha}-R^{\lambda}_{\cdot\mu\nu\omega/\alpha}\xi^{\alpha}{}_{;\beta}\dot{x}^{\beta}\\&-R^{\lambda}_{\cdot\mu\nu\omega}\xi^{\lambda}{}_{;\alpha}+R^{\lambda}_{\cdot\alpha\nu\omega}\xi^{\alpha}{}_{;\mu}+R^{\lambda}_{\cdot\mu\alpha\omega}\xi^{\alpha}{}_{;\nu}+R^{\lambda}_{\cdot\mu\nu\alpha}\xi^{\alpha}{}_{;\omega}]dt\end{aligned}$$

by virtue of the identities (1.11), (1.13) and (1.14). The equation (5.13) shows the validity of the equation (5.11).