3. *On the Flat Conformal Differential Geometry, III.*[1]

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§3. *Theory of curves.* (*Continued*)

In Paragraph 1 of the present Chapter, we have established the Frenet formulae (3.20) for a curve in a flat conformal space C_n. But, the parameter t adopted there being defined by a Schwarzian differential equation, it is determined only up to a homographic transformation. The Frenet formulae (3.20) are not invariant under this homographic transfomation of the parameter t. Consequently, the curvatures $\kappa_{(1)}, \kappa_{(2)}, \ldots\ldots, \kappa_{(n-1)}$ appearing there are not conformal quantities attached intrinsically to the curve.

In the next Paragraph, we shall introduce a purely conformal parameter σ on a curve, and establish the purely conformal Frenet formulae with respect to this conformal parameter σ.

2°. *The Frenet formulae with respect to a conformal parameter.*

Let us consider a homographic transformation

$$\bar{t} = \frac{at+b}{ct+d} \qquad (ad-bc \neq 0) \tag{3.24}$$

of the projective parameter t. Then, the current point-hypersphere $S_{(0)}$ defined by (3.4) is transformed into

$$\bar{S}_{(0)} = \frac{ad-bc}{(ct+d)^2} S_{(0)}, \tag{3.25}$$

the unit hypersphere $S_{(1)}$ defined by (3.2) into

$$\bar{S}_{(1)} = -\frac{2c}{ct+d} S_{(0)} + S_{(1)}, \tag{3.26}$$

and the point-hypersphere $S_{(\infty)}$ defined by (3.6) into

$$\bar{S}_{(\infty)} = +\frac{2c^2}{ad-bc} S_{(0)} - \frac{2c(ct+d)}{ad-bc} S_{(1)} + \frac{(ct+d)^2}{ad-bc} S_{(\infty)}. \tag{3.27}$$

Thus, if we put

$$\frac{d}{d\bar{t}} \bar{S}_{(\infty)} = \bar{\kappa}_{(1)} \bar{S}_{(2)}, \tag{3.28}$$

we obtain, from (3.27),

$$\bar{\kappa}_{(1)} \bar{S}_{(2)} = \frac{(ct+d)^4}{(ad-bc)^2} \kappa_{(1)} S_{(2)}.$$

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1) Cf. K. Yano: On the flat conformal differential geometry I, II. Proc. *21* (1945), 419–429; 454–465.

The $\bar{S}_{(2)}$ and $S_{(2)}$ being both unit hyperspheres, we find

$$\bar{\kappa}_{(1)} = \frac{(ct+d)^4}{(ad-bc)^2}\kappa_{(1)},$$

or

$$\kappa_{(1)} = \left(\frac{dt}{d\bar{t}}\right)^2\kappa_{(1)},$$

from which

$$(\bar{\kappa}_{(1)})^{\frac{1}{2}}d\bar{t} = (\kappa_{(1)})^{\frac{1}{2}}dt. \tag{3.29}$$

Consequently, the differential defined by

$$d\sigma = (\kappa_{(1)})^{\frac{1}{2}}dt \tag{3.30}$$

is invariant under any homographic transformation of the projective parameter. The parameter defined by $\sigma = \int d\sigma$ may be called conformal parameter(1) on the curve. For a circle, the first curvature $\kappa_{(1)}$ vanishes, and consequently the conformal parameter does not exist. It plays the rôle of a minimal curve in the conformal geometry.

Substituting

$$\kappa_{(1)} = k_{(1)} / \dot{t}^2 \text{ and } k_{(1)}^2 = g_{\mu\nu}v^\mu v^\nu$$

in the expression of σ, we obtain

$$\sigma = \int (g_{\mu\nu}v^\mu v^\nu)^{\frac{1}{4}}ds, \tag{3.31}$$

where the vector v^λ is defined by (3.12).

The conformal arc length σ being thus defined, the point-hypersphere

$$R_{(0)} = \dot{\sigma}A_0 \tag{3.32}$$

is a conformal current point on the curve.

Differentiating (3.32) along the curve, we find

$$R_{(1)} = \frac{d}{d\sigma}R_{(0)} = \frac{\ddot{\sigma}}{\dot{\sigma}}A_0 + \frac{d\xi^\lambda}{ds}A_\lambda, \tag{3.33}$$

which is also conformal unit hypersphere orthogonal to the curve.

Differentiating the equation (3.33) along the curve, we find

$$\frac{d}{d\sigma}R_{(1)} = \frac{1}{\dot{\sigma}}\left[\left(\frac{\dddot{\sigma}}{\dot{\sigma}} - \frac{\ddot{\sigma}^2}{\dot{\sigma}^2} + a^0\right)A_0 + \left(\frac{\delta^2\xi^\lambda}{ds^2} + \frac{\ddot{\sigma}}{\dot{\sigma}}\frac{d\xi^\lambda}{ds}\right)A_\lambda + A_\infty\right].$$

The hypershere $\frac{d}{d\sigma}R_{(1)}$ being not in general a point-hypersphere, we shall seek for a function $\lambda_{(1)}$ such that

$$\frac{d}{d\sigma}R_{(1)} - \lambda_{(1)}R_{(0)} = \frac{1}{\dot{\sigma}}\left[\left(\frac{\dddot{\sigma}}{\dot{\sigma}} - \frac{\ddot{\sigma}^2}{\dot{\sigma}^2} + a^0 - \dot{\sigma}^2\lambda_{(1)}\right)A_0 + \left(\frac{\delta^2\xi^\lambda}{ds^2} + \frac{\ddot{\sigma}}{\dot{\sigma}}\frac{d\xi^\lambda}{ds}\right)A_\lambda + A_\infty\right] \tag{3.34}$$

be a point-hypersphere. In order that it will be the case, we must have

1) K. Yano and Y. Mutô: On the conformal arc length. Proc. *17* (1941), 318–322.

$$g_{\mu\nu}a^\mu a^\nu + \frac{\ddot{\sigma}^2}{\dot{\sigma}^2} - 2\left(\frac{\dddot{\sigma}}{\dot{\sigma}} - \frac{\ddot{\sigma}^2}{\dot{\sigma}^2} + a^0 - \dot{\sigma}^2\lambda_{(1)}\right) = 0,$$

or

(3.35) $$\lambda_{(1)} = \frac{1}{\dot{\sigma}^2}\left[\{\sigma, s\} - (\frac{1}{2}g_{\mu\nu}a^\mu a^\nu - a^0)\right].$$

The $\lambda_{(1)}$ thus defined is a purely conformal curvature of the curve.

If we put

$$\frac{d}{d\sigma}R_{(1)} - \lambda_{(1)}R_{(0)} = R_{(\infty)},$$

or

(3.36) $$\frac{d}{d\sigma}R_{(1)} = \lambda_{(1)}R_{(0)} + R_{(\infty)},$$

the $R_{(\infty)}$ is a point-hypersphere on the unit hypersphere $R_{(1)}$ and satisfy

$$R_{(0)}R_{(\infty)} = -1.$$

Substituting (3.35), and (3.36) into (3.34), we find

(3.37) $$R_{(\infty)} = \frac{1}{\dot{\sigma}}\left[\frac{1}{2}\left(\frac{\ddot{\sigma}^2}{\dot{\sigma}^2} + g_{\mu\nu}a^\mu a^\nu\right)A_0 + \left(a^\lambda + \frac{\ddot{\sigma}}{\dot{\sigma}}\frac{d\xi^\lambda}{ds}\right)A_\lambda + A_\infty\right].$$

Now, differentiating the relations

$$R_{(0)}R_{(\infty)} = -1, \quad R_{(1)}R_{(\infty)} = 0, \quad R_{(\infty)}R_{(\infty)} = 0$$

along the curve, we know that $-\lambda_{(1)}R_{(1)} + \frac{d}{d\sigma}R_{(\infty)}$ is a hypersphere passing through the points $R_{(0)}$ and $R_{(\infty)}$ and orthogonal to the unit hypersphere $R_{(1)}$. On the other hand, we have, from (3.37),

$$-\lambda_{(1)}R_{(1)} + \frac{d}{d\sigma}R_{(\infty)} = \frac{1}{\dot{\sigma}^2}\left[g_{\mu\nu}\frac{\delta a^\mu}{\delta s}a^\nu + \Pi^0_{\mu\nu}a^\mu\frac{d\xi^\nu}{ds}\right]A_0$$
$$+ \frac{1}{\dot{\sigma}^2}\left[\frac{\delta a^\lambda}{ds} + (g_{\mu\nu}a^\mu a^\nu - a^0)\frac{d\xi^\lambda}{ds} + \Pi^\lambda_{\infty\nu}\frac{d\xi^\nu}{ds}\right]A_\lambda,$$

or

(3.38) $$-\lambda_{(1)}R_{(1)} + \frac{d}{d\sigma}R_{(\infty)} = S_{(2)},$$

which shows that the hypersphere $S_{(2)}$ is invariant under homographic transformation of the projective parameter t. Thus, if we put $R_{(2)} = S_{(2)}$, we have

(3.39) $$\frac{d}{d\sigma}R_{(\infty)} = \lambda_{(1)}R_{(1)} + R_{(2)},$$

where

(3·40) $$R_{(2)} = \frac{1}{\dot{\sigma}^2}[v^0A_0 + v^\lambda A_\lambda]$$

is a unit hypersphere passing through the points $R_{(0)}$ and $R_{(\infty)}$ and being orthogonal to the hypersphere $R_{(1)}$.

Now, differentaiting the relations

$$R_{(0)}R_{(2)} = 0, \quad R_{(1)}R_{(2)} = 0, \quad R_{(\infty)}R_{(2)} = 0, \quad R_{(2)}R_{(2)} = 1$$

with respect to σ, we find that $-R_{(0)} + \frac{d}{d\sigma}R_{(2)}$ is a hypersphere passing through the points $R_{(0)}$ and $R_{(\infty)}$ and being orthogonal to $R_{(1)}$ and $R_{(2)}$.

Thus we can put

$$-R_{(0)} + \frac{d}{d\sigma} R_{(2)} = \lambda_{(2)} R_{(3)},$$

or

$$\frac{d}{d\sigma} R_{(2)} = R_{(0)} + \lambda_{(2)} R_{(3)}, \tag{3.41}$$

where $R_{(3)}$ is a unit hypersphere passing through $R_{(0)}$ and $R_{(\infty)}$ and orthogonal to $R_{(1)}$ and $R_{(2)}$.

Comparing the equation (3.41) with (3.16) and remembering that $R_{(0)} = \frac{d\sigma}{dt} S_{(0)}$, $R_{(2)} = S_{(2)}$ and $\kappa_{(1)} = \left(\frac{d\sigma}{dt}\right)^2$, we find

$$R_{(3)} = S_{(3)} \text{ and } \lambda_{(2)} = \kappa_{(2)} \Big/ \frac{d\sigma}{dt}. \tag{3.42}$$

Putting (3.40) in the form

$$R_{(2)} = \overset{0}{\eta}_{(2)} A_0 + \overset{\lambda}{\eta}_{(2)} A_\lambda, \tag{3.43}$$

where

$$\overset{0}{\eta}_{(2)} = \frac{v^0}{\dot{\sigma}^2}, \quad \overset{\lambda}{\eta}_{(2)} = \frac{v^\lambda}{\dot{\sigma}^2}, \tag{3.44}$$

we have, by differentiation,

$$\frac{d}{d\sigma} R_{(2)} = \left(\frac{d\overset{0}{\eta}_{(2)}}{d\sigma} + \Pi^0_{\mu\nu} \overset{\mu}{\eta}_{(2)} \overset{\nu}{\eta}_{(1)}\right) A_0 + \left(\frac{\delta \overset{\lambda}{\eta}_{(2)}}{d\sigma} + \overset{0}{\eta}_{(2)} \overset{\lambda}{\eta}_{(1)}\right) A_\lambda$$

and consequently

$$-R_{(0)} + \frac{d}{d\sigma} R_{(2)} = \left(\frac{d\overset{0}{\eta}_{(2)}}{d\sigma} + \Pi^0_{\mu\nu} \overset{\mu}{\eta}_{(2)} \overset{\nu}{\eta}_{(1)} - \dot{\sigma}\right) A_0 + \left(\frac{\delta \overset{\lambda}{\eta}_{(2)}}{d\sigma} + \overset{0}{\eta}_{(2)} \overset{\lambda}{\eta}_{(1)}\right) A_\lambda$$

by virtue of the relations

$$g_{\mu\nu} \overset{\mu}{\eta}_{(2)} \overset{\nu}{\eta}_{(1)} = 0, \quad \overset{\lambda}{\eta}_{(1)} = \frac{d\xi^\lambda}{d\sigma}.$$

Thus, putting

$$R_{(3)} = \overset{0}{\eta}_{(3)} A_0 + \overset{\lambda}{\eta}_{(3)} A_\lambda, \tag{3.45}$$

we have

$$\begin{cases} \dfrac{d\overset{0}{\eta}_{(2)}}{d\sigma} + \Pi^0_{\mu\nu} \overset{\mu}{\eta}_{(2)} \overset{\nu}{\eta}_{(1)} = \dot{\sigma} + \overset{0}{\eta}_{(3)}, \\ \dfrac{\delta \overset{\lambda}{\eta}_{(2)}}{d\sigma} + \overset{0}{\eta}_{(2)} \eta_{(1)} = \lambda_{(2)} \eta_{(3)}, \end{cases} \tag{3.46}$$

where $\overset{\lambda}{\eta}_{(3)}$ is a unit vector orthogonal to $\overset{\lambda}{\eta}_{(1)}$ and $\overset{\lambda}{\eta}_{(2)}$.

Differentiating next the relations

$$R_{(0)} R_{(3)} = 0, \quad R_{(1)} R_{(3)} = 0, \quad R_{(\infty)} R_{(3)} = 0, \quad R_{(2)} R_{(3)} = 0, \quad R_{(3)} R_{(3)} = 1$$

with respect to σ, we find that $\lambda_{(2)} R_{(2)} + \frac{d}{d\sigma} R_{(3)}$ is a hypersphere passing through the points $R_{(0)}$ and $R_{(\infty)}$ and being orthogonal to $R_{(1)}$, $R_{(2)}$ and $R_{(3)}$. Thus we can put

$$\lambda_{(2)} R_{(2)} + \frac{d}{d\sigma} R_{(3)} = \lambda_{(3)} R_{(4)}$$

or

$$\frac{d}{d\sigma}R_{(3)} = -\lambda_{(2)}R_{(2)} + \lambda_{(3)}R_{(4)}, \tag{3.47}$$

where $R_{(4)}$ is a unit hypersphere passing through $R_{(0)}$ and $R_{(\infty)}$ and orthogonal to $R_{(1)}$, $R_{(2)}$ and $R_{(3)}$.

Comparing (3.47) with (3.18), we find

$$R_{(4)} = S_{(4)} \quad \text{and} \quad \lambda_{(3)} = \kappa_{(3)} \Big/ \frac{d\sigma}{dt}. \tag{3.48}$$

If we put

$$R_{(4)} = \eta^{0}_{(4)}A_0 + \eta^{\lambda}_{(4)}A_\lambda, \tag{3.49}$$

we have, from (3.45) and (3.47),

$$\begin{cases} \dfrac{d\eta^{0}_{(3)}}{d\sigma} + \Pi^{0}_{\mu\nu}\eta^{\mu}_{(3)}\eta^{\nu}_{(1)} = -\lambda_{(2)}\eta^{0}_{(2)} + \lambda_{(3)}\eta^{0}_{(4)}, \\ \dfrac{\delta\eta^{\lambda}_{(3)}}{d\sigma} + \eta^{0}_{(3)}\eta^{\lambda}_{(1)} = -\lambda_{(2)}\eta^{\lambda}_{(2)} + \lambda_{(3)}\eta^{\lambda}_{(4)}, \end{cases} \tag{3.50}$$

where $\eta^{\lambda}_{(4)}$ is a unit vector orthogonal to $\eta^{\lambda}_{(1)}$, $\eta^{\lambda}_{(2)}$ and $\eta^{\lambda}_{(3)}$.

Proceeding in this manner, we shall arrive at the formulae

$$\frac{d}{d\sigma}R_{(0)} = R_{(1)}, \quad \frac{d}{d\sigma}R_{(1)} = \lambda_{(1)}R_{(0)} + R_{(\infty)}, \quad \frac{d}{d\sigma}R_{(\infty)} = \lambda_{(1)}R_{(1)} + R_{(2)},$$

$$\begin{cases} \dfrac{d}{d\sigma}R_{(2)} = R_{(0)} + \lambda_{(2)}R_{(3)}, \\ \dfrac{d}{d\sigma}R_{(3)} = -\lambda_{(2)}R_{(2)} + \lambda_{(3)}R_{(4)}, \\ \dfrac{d}{d\sigma}R_{(4)} = -\lambda_{(3)}R_{(3)} + \lambda_{(4)}R_{(5)}, \\ \cdots\cdots\cdots\cdots \\ \dfrac{d}{d\sigma}R_{(n)} = -\lambda_{(n-1)}R_{(n-1)}, \end{cases} \tag{3.51}$$

where $R_{(1)}, R_{(2)}, \ldots\ldots, R_{(n)}$ are n mutually orthogonal unit hyperspheres all passing through the points $R_{(0)}$ and $R_{(\infty)}$, $R_{(0)}$ being a point on the curve and $R_{(1)}$ a unit hypersphere orthogonal to the curve.

These are the Frenet formulae for the curve with respect to a conformal arc length σ.

If we put

$$\begin{cases} R_{(0)} = \dot{\sigma}A_0, \quad R_{(1)} = \dfrac{\ddot{\sigma}}{\dot{\sigma}}A_0 + \dfrac{d\xi^\lambda}{ds}A_\lambda, \\ R_{(\infty)} = \dfrac{1}{\dot{\sigma}}\left[\dfrac{1}{2}\left(\dfrac{\ddot{\sigma}^2}{\dot{\sigma}^2} + g_{\mu\nu}a^\mu a^\nu\right)A_0 + \left(a^\lambda + \dfrac{\ddot{\sigma}}{\dot{\sigma}}\dfrac{d\xi^\lambda}{ds}\right)A_\lambda + A_\infty\right], \\ R_{(2)} = \eta^{0}_{(2)}A_0 + \eta^{\lambda}_{(2)}A_\lambda, \\ R_{(3)} = \eta^{0}_{(3)}A_0 + \eta^{\lambda}_{(3)}A_\lambda, \\ \cdots\cdots\cdots\cdots \\ R_{(n)} = \eta^{0}_{(n)}A_0 + \eta^{\lambda}_{(n)}A_\lambda, \end{cases} \tag{3.52}$$

we have

$$(3.53)\quad \begin{cases} \dfrac{d\eta^0_{(2)}}{d\sigma} + \Pi^0_{\mu\nu}\eta^\mu_{(2)}\eta^\nu_{(1)} = \dot{\sigma} + \eta^0_{(3)}, \\ \dfrac{d\eta^0_{(3)}}{d\sigma} + \Pi^0_{\mu\nu}\eta^\mu_{(3)}\eta^\nu_{(1)} = -\lambda_{(2)}\eta^0_{(2)} + \lambda_{(3)}\eta^0_{(4)}, \\ \dfrac{d\eta^0_{(4)}}{d\sigma} + \Pi^0_{\mu\nu}\eta^\mu_{(4)}\eta^\nu_{(1)} = -\lambda_{(3)}\eta^0_{(3)} + \lambda_{(4)}\eta^0_{(5)}, \\ \cdots\cdots\cdots\cdots \\ \dfrac{d\eta^0_{(n-1)}}{d\sigma} + \Pi^0_{\mu\nu}\eta^\mu_{(n-1)}\eta^\nu_{(1)} = -\lambda_{(n-2)}\eta^0_{(n-2)} + \lambda_{(n-1)}\eta^0_{(n)}, \\ \dfrac{d\eta^0_{(n)}}{d\sigma} + \Pi^0_{\mu\nu}\eta^\mu_{(n)}\eta^\nu_{(1)} = -\lambda_{(n-1)}\eta^0_{(n-1)}, \end{cases}$$

and

$$(3.54)\quad \begin{cases} \dfrac{\delta\eta^\lambda_{(2)}}{d\sigma} + \eta^0_{(2)}\eta^\lambda_{(1)} = \lambda_{(2)}\eta^\lambda_{(3)}, \\ \dfrac{\delta\eta^\lambda_{(3)}}{d\sigma} + \eta^0_{(3)}\eta^\lambda_{(1)} = -\lambda_{(2)}\eta^\lambda_{(2)} + \lambda_{(3)}\eta^\lambda_{(4)}, \\ \dfrac{\delta\eta^\lambda_{(4)}}{d\sigma} + \eta^0_{(4)}\eta^\lambda_{(1)} = -\lambda_{(3)}\eta^\lambda_{(3)} + \lambda_{(4)}\eta^\lambda_{(5)}, \\ \cdots\cdots\cdots\cdots \\ \dfrac{\delta\eta^\lambda_{(n-1)}}{d\sigma} + \eta^0_{(n-1)}\eta^\lambda_{(1)} = -\lambda_{(n-2)}\eta^\lambda_{(n-2)} + \lambda_{(n-1)}\eta^\lambda_{(n)}, \\ \dfrac{\delta\eta^\lambda_{(n)}}{d\sigma} + \eta^0_{(n)}\eta^\lambda_{(1)} = -\lambda_{(n-1)}\eta^\lambda_{(n-1)}. \end{cases}$$

These are purely comformal Frenet formulae for the curve with respect to a conformal parameter σ.

3°. *Method of E. Cartan.*

In Paragraph 1 of the present Chapter, we have established the Frenet formulae for a curve in the conformal space by the use of a projective parameter t and in Paragraph 2, we have introduced a conformal parameter σ on the curve, and we have modified the Frenet formulae of Paragraph 1 so as to be purely conformal, that is to say, to be independent of the choice of the projective parameter t.

In the present Paragraph, we shall show how we can obtain directly the conformal Frenet formulae applying the method of repère mobile of E. Cartan.

Let us consider a curve $\xi^\lambda(\gamma)$ in the conformal space C_n and attach, at eacn point of the curve, a repere mobile $[R_0, R_1, R_2, \ldots, R_n, R_\infty]$, where R_0 is a point-hypersphere coinciding with the current point of the curve, $R_1, R_2, \ldots, R_n$, n mutually orthogonal hyperspheres passing through the point R_0 and finally R_∞ the point of intersection other than R_0 of n hyperspheres R_1, $R_2, \ldots, R_n$ such that $R_0 R_\infty = -1$.

Then we have, along the curve, formulae of the form

$$(3.55)\quad \begin{cases} dR_0 = \omega_{00}R_0 + \omega_{01}R_1 + \omega_{02}R_2 + \dots\dots\dots\dots + \omega_{0n}R_n + \omega_{0\infty}R_\infty, \\ dR_1 = \omega_{10}R_0 + \omega_{11}R_1 + \omega_{12}R_2 + \dots\dots\dots\dots + \omega_{1n}R_n + \omega_{1\infty}R_\infty, \\ dR_2 = \omega_{20}R_0 + \omega_{21}R_1 + \omega_{22}R_2 + \dots\dots\dots\dots + \omega_{2n}R_n + \omega_{2\infty}R_\infty, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ dR_n = \omega_{n0}R_0 + \omega_{n1}R_1 + \omega_{n2}R_2 + \dots\dots\dots\dots + \omega_{nn}R_n + \omega_{n\infty}R_\infty, \\ dR_\infty = \omega_{\infty 0}R_0 + \omega_{\infty 1}R_1 + \omega_{\infty 2}R_2 + \dots\dots\dots\dots + \omega_{\infty n}R_n + \omega_{\infty\infty}R_\infty, \end{cases}$$

where ω s are Pfaffian forms with respect to the principal parameter r which determines the position of the point R_0 on the curve, and the secondary parameters $u_1, u_2, u_3, \dots\dots$ which determines the position of the repère mobile $[R_0, R_1, R_2, \dots, R_n, R_\infty]$.

The inner products $R_0R_0=0$, $R_0R_\lambda = 0$, $R_0R_\infty = -1$, $R_\lambda R_\mu = \delta_{\lambda\mu}$, $R_\lambda R_\infty = 0$, and $R_\infty R_\infty = 0$ being all constants, we must have

$$(3.56)\quad \begin{cases} \omega_{00} + \omega_{\infty\infty} = 0, \quad \omega_{0\infty} = \omega_{\infty 0} = 0, \\ \omega_{\infty\lambda} = \omega_{\lambda 0}, \quad \omega_{\lambda\infty} = \omega_{0\lambda}, \quad \omega_{\mu\nu} + \omega_{\nu\mu} = 0. \end{cases}$$

Consequently, the formulae (3.55) take the form

$$(3.57)\quad \begin{cases} dR_0 = \omega_{00}R_0 + \omega_{01}R_1 + \omega_{02}R_2 + \dots\dots\dots\dots + \omega_{0n}R_n, \\ dR_1 = \omega_{10}R_0 \qquad\qquad + \omega_{12}R_2 + \dots\dots\dots\dots + \omega_{1n}R_n + \omega_{01}R_\infty, \\ dR_2 = \omega_{20}R_0 - \omega_{12}R_1 \qquad\qquad + \dots\dots\dots\dots + \omega_{2n}R_n + \omega_{02}R_\infty, \\ \dots\dots\dots\dots\dots\dots \\ dR_n = \omega_{n0}R_0 - \omega_{1n}R_1 - \omega_{2n}R_2 - \dots\dots\dots\dots \qquad + \omega_{0n}R_\infty, \\ dR_\infty = \qquad \omega_{10}R_1 + \omega_{20}R_2 + \dots\dots\dots\dots + \omega_{n0}R_n - \omega_{00}R_\infty. \end{cases}$$

We observe here that the Pfaffians $\omega_{0\lambda}$ do not contain the differentials of the secondary parameters, for, if we fix the principal parameter r, then the point R_0 is fixed and consequently $\omega_{0\lambda} = 0$.

Moreover, the Pfaffians ω's must satisfy the structure equations

$$(3.58)\quad \begin{cases} (\omega_{00})' = \sum_{\nu=1}^{n}[\omega_{0\nu}\,\omega_{\nu 0}], \\ (\omega_{0\lambda})' = [\omega_{00}\,\omega_{0\lambda}] + \sum_{\nu=1}^{n}[\omega_{0\nu}\,\omega_{\nu\lambda}], \\ (\omega_{\lambda\mu})' = [\omega_{\lambda 0}\,\omega_{0\mu}] + [\omega_{0\lambda}\,\omega_{\mu 0}] + \sum_{\nu=1}^{n}[\omega_{\lambda\nu}\,\omega_{\nu\mu}], \\ (\omega_{\lambda 0})' = [\omega_{\lambda 0}\,\omega_{00}] + \sum_{\nu=1}^{n}[\omega_{\lambda\nu}\,\omega_{\nu 0}], \end{cases}$$

where the prime denotes the outer derivative and the parenthesis the outer product of the Pfaffians.

We shall now choose the unit hypersphere R_1 in such a way that R_1 should be orthogonal to the curve. Then R_1 belonging to the pencil of hyperspheres determined by R_0 and $R_0 + dR_0$, we have

$$(3.59)\qquad dR_0 = \omega_{00}R_0 + \omega_{01}R_1$$

and

(3.60) $$\boxed{\omega_{0i} = 0} \qquad (i=2, 3, \ldots\ldots, n).$$

Here ω_{01} is not identically zero, for if $\omega_{01} \equiv 0$, the point R_0 will be always fixed. We shall call such a repère the repère mobile of the first order.

Differentiating (3.60) exteriorly and taking account of (3.60) itself, we find

$$[\omega_{01}\,\omega_{1i}] = 0.$$

Following a lemma of E. Cartan, we have, from the above equation,

(3.61) $$\omega_{1i} = a_i\,\omega_{01}.$$

Differentiating this equation exteriorly and taking account of (3.58), (3.60) and (3.61), we find

$$[d\,a_i + a_i\,\omega_{00} + \sum_{a=2}^{n}{}' a_a\,\omega_{ai} + \omega_{i0},\ \omega_{01}] = 0,$$

from which

(3.62) $$d\,a_i + a_i\,\omega_{00} + \sum_{a=2}^{n} a_a\,\omega_{ai} + \omega_{i0} = \beta_i\,\omega_{01}.$$

Thus, if we fix the principal parameter r and vary only the secondary parameters, we have

$$\delta\,a_i + a_i\,e_{00} + \sum_{a=2}^{n} a_a\,e_{ai} + e_{i0} = 0,$$

where δ denotes the differential with respect to the variations of the secondary parameters, and $\omega(\delta) = e$.

The above equations show that we can arrange the secondary parameters in such a way that we have $a_i = 0$.

If we perform this specialization of the repère mobile, we find, from (3.61) and (3.62),

(3.63) $$\omega_{1i} = 0$$

and

(3.64) $$\omega_{i0} = \beta_i\,\omega_{01}$$

respectively. We shall call the repère mobile of the second order, a repère mobile whose relative components ω satisfy

(3.65) $$\boxed{\omega_{0i} = 0, \quad \omega_{1i} = 0} \qquad (i = 2, 3, \ldots\ldots, n).$$

Differentiating (3.64) exteriorly and taking account of (3.58), (3.64) and (3.65) we find

$$[d\,\beta_i + 2\,\beta_i\,\omega_{00} + \sum_{a=2}^{n}{}' \beta_a\,\omega_{ai},\ \omega_{01}] = 0,$$

from which

(3.66) $$d\,\beta_i + 2\,\beta_i\,\omega_{00} + \sum_{a=2}^{n} \beta_a\,\omega_{ai} = \gamma_i\,\omega_{01}.$$

Thus if we fix the principal parameter r, we find

$$\delta \beta_i + 2 \beta_i e_{00} + \sum_{a=2}^{n} \beta_a e_{ai} = 0$$

for the variations of the secondary parameters.

Here, we must consider two cases according as $\beta_i \equiv 0$ or $\beta_i \not\equiv 0$.

If $\beta_i \equiv 0$, we have $\omega_{i0} = 0$ from (3.64). Thus our formulae (3.57) take the form

$$(3.67) \qquad \begin{cases} dR_0 = \omega_{00} R_1 + \omega_{01} R_1, \\ dR_1 = \omega_{10} R_0 + \omega_{01} R_\infty, \\ dR_\infty = \omega_{10} R_1 - \omega_{00} R_\infty. \end{cases}$$

Substituting here R_1 by a hypersphere of the form $a R_0 + R_1$ and R_∞ by $\frac{1}{2} a^2 R_0 + a R_1 + R_\infty$, we can put (3.67) in the form

$$\begin{cases} dR_0 = \omega_{00} R_0 + \omega_{01} R_1, \\ dR_1 = \omega_{01} R_\infty, \\ dR_\infty = - \omega_{00} R_\infty. \end{cases}$$

Then, multiplying R_0 and R_1 by a suitable factor and dividing R_∞ by the same factor, we obtain finally

$$(3.68) \qquad \begin{cases} dR_0 = \omega_{01} R_1, \\ dR_1 = \omega_{01} R_\infty, \\ dR_\infty = 0. \end{cases}$$

If we put $\omega_{01} = dt$, we have

$$\frac{d}{dt} R_0 = R_1, \quad \frac{d}{dt} R_1 = R_\infty, \quad \frac{d}{dt} R_\infty = 0,$$

thus, the curves for which $\beta_i \equiv 0$ are circles of the conformal space C_n.

Let us return to the general case $\beta_i \not\equiv 0$. In this case, the equations which give $\delta \beta_i$ show that we can arrange the secondary parameters in such a way that we have $\beta_2 = 1$ and $\beta_j = 0$ $(j = 3, 4, \ldots\ldots, n)$.

Then we have, from (3.64) and (3.66),

$$(3.69) \qquad \omega_{20} = \omega_{01},$$

$$(3.70) \qquad \omega_{j0} = 0 \qquad (j=3, 4, \ldots\ldots, n)$$

and

$$(3.71) \qquad 2\omega_{00} = \gamma_2 \omega_{01}, \qquad \omega_{2j} = \gamma_j \omega_{01}$$

respectively.

We shall call the repère mobile of the third order a repere mobile whose relative components satisfy the relations

$$(3.72) \qquad \boxed{\omega_{0i} = 0, \quad \omega_{1i} = 0, \quad \omega_{20} = \omega_{01}, \quad \omega_{j0} = 0} \qquad \begin{matrix} (i = 2, 3, \ldots, n), \\ (j = 3, 4, \ldots, n), \end{matrix}$$

the relations (3.71) being its consequences.

For a repère mobile of the third order, we have

$$(\dot{\omega}_{01})' = d\,\omega_{01}(\delta) - \delta\,\omega_{01}(d) = -\,\delta\,\omega_{01}(d) = 0,$$

which shows that ω_{01} is an intrinsic quantity of the curve. We shall call it differential of the conformal arc length and denote it by $d\sigma$.

Differentiating exteriorly the first of the equations (3.71), and taking account of the relations (3.58), (3.71) and (3.72), we fnd

$$[d\,\gamma_2 + \gamma_2\,\omega_{00} + 2\,\omega_{10},\ \omega_{01}] = 0,$$

consequently

(3.73) $$d\,\gamma_2 + \gamma_2\,\omega_{00} + 2\,\omega_{10} = \theta_2\,\omega_{01},$$

from which

$$\delta\,\gamma_2 + \gamma_2\,e_{00} + 2\,e_{10} = 0$$

for the variations of the secondary parameters. This equation shows that we can arrange the secondary parameters in such a way that we have $\gamma_2 = 0$, and consequently

(3.74) $$\omega_{00} = 0$$

and

(3.75) $$2\,\omega_{10} = \theta_2\,\omega_{01}.$$

Differentiating exteriorly the second of the equations (3.71), and taking account of the relations (3.58), (3.71), (3.72) and (3.73), we find

$$[d\gamma_j + \sum_{b=3}^{n} \gamma_b\,\omega_{bj},\quad \omega_{01}] = 0,$$

from which

(3.76) $$d\,\gamma_j + \sum_{b=3}^{n} \gamma_b\,\omega_{bj} = \theta_j\,\omega_{01}.$$

Consequently we have

$$\delta\,\gamma_j + \sum_{b=3}^{n} \gamma_b\,e_{bj} = 0$$

for the variations of the secondary parameters.

If $\gamma_j \equiv 0$, we have, from (3.71), $\omega_{2j} = 0$, and the formulae (3.57) take the form

(3.77) $$\begin{cases} d\,R_0 = \omega_{01}\,R_1, \\ d\,R_1 = \omega_{10}\,R_1 + \omega_{01}\,R_\infty, \\ d\,R_\infty = -\,\omega_{10}\,R_1 + \omega_{01}\,R_2, \\ d\,R_2 = \omega_{01}\,R_0, \end{cases}$$

thus, the curve is a curve on a two-dimensional sphere.

If $\gamma_j \not\equiv 0$, the equations giving $\delta\gamma_j$ show that we can arrange the secondary parameters in such a way that we have $\gamma_k = 0$ $(k = 4, 5, \ldots\ldots, n)$, and consequently

(3.78) $$\omega_{2k} = 0,$$

(3.79) $$\omega_{23} = \gamma_3\,\omega_{01},$$

and

(3.80) $$\omega_{3k} = \theta_k \, \omega_{01}.$$

We shall call the repère mobile of the fourth order a repère mobile whose relative components satisfy the relations

(3.81) $$\omega_{0i} = 0, \quad \omega_{1i} = 0, \quad \omega_{20} = \omega_{01}, \quad \omega_{j0} = 0, \quad \omega_{00} = 0, \quad \omega_{2k} = 0$$

$$(i = 2, 3, \ldots, n; \; j = 3, 4, \ldots, n; \; k = 4, 5, \ldots, n),$$

relations (3.75) and (3.79) and (3.80) being its consequences.

For a repère mobile of the fourth order, we have

$$(\omega_{10})' = d\,\omega_{10}(\delta) - \delta\,\omega_{10}(d) = 0.$$

But, the equation (3.75) shows that $\omega_{10}(\delta) = \frac{1}{2}\theta_2\,\omega_{01}(\delta) = 0$, and consequently

$$\delta\,\omega_{10}(d) = 0.$$

Thus, ω_{10} is an intrinsic quantity of the curve, so we shall put

(3.82) $$\frac{\omega_{10}}{\omega_{01}} = \lambda_1,$$

and call it the first conformal curvature of the curve.

For such a repère, we have also

$$(\omega_{23})' = d\,\omega_{23}(\delta) - \delta\,\omega_{23}(d) = 0.$$

But, the equation (3.79) shows that $\omega_{23}(\delta) = \gamma_3\,\omega_{01}(\delta) = 0$, and consequently

$$\delta\,\omega_{23}(d) = 0.$$

Thus ω_{23} is an intrinsic quantity of the curve, and consequently we shall put

(3.83) $$\frac{\omega_{23}}{\omega_{01}} = \lambda_2,$$

and call it the second conformal curvature of the curve.

Continuing in this way, we shall arrive at the formulae

(3.84) $$\begin{cases} dR_0 = & d\sigma\,R_1, \\ dR_1 = \lambda_1\,d\sigma\,R_1 + d\sigma\,R_\infty, \\ dR_\infty = \lambda_1\,d\sigma\,R_1 + d\sigma\,R_2, \\ dR_2 = d\sigma\,R_0 + \lambda_2\,d\sigma\,R_3, \\ dR_3 = -\lambda_3\,d\sigma\,R_3 + \lambda_4\,d\sigma\,R_4, \\ \qquad \cdots\cdots\cdots\cdots \\ dR_n = -\lambda_{n-1}\,d\sigma\,R_{n-1}, \end{cases}$$

which coincide with (3.51).

The quantities $d\sigma$, λ_1, λ_2, ..., λ_{n-1} appearing in these formulae being purely conformal invariants, we can develop here the theory of natural equations for a curve in the conformal space C_n.[1]

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1) A. Fialkow: The conformal theory of curves. Proc. Nat. Acad. Sci. U.S.A., *26* (1940), 437–439.