19. Note on Eulerean Squares.

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1. By an Eulerean square of order n we mean a matrix $\|(a_{ij}, b_{ij})\|$, $i, j = 1, 2 \ldots\ldots, n$, formed by n^2 pairs (a_{ij}, b_{ij}) out of n symbols, say, $1, 2, \ldots\ldots, n$, so arranged that neither in a row nor in a column of the matrix one and the same symbol occurs more than once as the first term a or as the second term b of the constituent (a, b), so that the matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ are the so-called Latin squares. Without loss of generality, we may assume an Eulerean square in the *normal* form, one in which the first row is made up of the constituents (i, i), $i = 1, 2, \ldots\ldots, n$.

The substitutions A_i, $i = 1, 2, \ldots\ldots, n$ of the lst. by the ith. row of a Latin square form what we call a *Latin system* of substitutions, which is characterized by the occurrence among them of all the n^2 *substitution-elements* $a \to b$, $a, b = 1, 2, \ldots\ldots, n$. We have then in connection with an Eulerean square in the normal form E, beside the *anterior* and the *posterior* Latin systems:

$$A_i = \begin{pmatrix} 1, & 2, & \ldots\ldots, & n \\ a_{i1}, & a_{i2}, & \ldots\ldots, & a_{in} \end{pmatrix}, \quad B_i = \begin{pmatrix} 1, & 2, & \ldots\ldots, & n \\ b_{i1}, & b^{i2}, & \ldots\ldots, & b_{in} \end{pmatrix},$$

the *intermediate* system

$$P_i = \begin{pmatrix} a_{i1}, & a_{i2}, & \ldots\ldots, & a_{in} \\ b_{i1}, & b_{i2}, & \ldots\ldots, & b_{in} \end{pmatrix}, \qquad i = 1, 2, \ldots\ldots, n,$$

which also make up a Latin system. Between the substitutions of the systems A, B and P the relation (i) subsists, whence also, observing that A_i^{-1} form with A_i a Latin system, the relations (ii) – (vi):

$$\begin{array}{lll} \text{(i) } A_i P_i = B_i, & \text{(ii) } B_i^{-1} A_i = P_i^{-1}, & \text{(iii) } A_i^{-1} B_i = P_i, \\ \text{(iv) } B_i P_i^{-1} = A_i, & \text{(v) } P_i B_i^{-1} = A_{i,}^{-1} & \text{(vi) } P_i^{-1} A_i^{-1} = B_i^{-1}, \end{array} \qquad (1)$$

shewing that the distinction between the extreme and the intermediate systems of an Eulerean square is relevant only to the mutual relation, not inherent in the nature of the systems themselves. Beside these, we have a *transverse* system Q consisting of the substitutions

$$Q_j = \begin{pmatrix} a_{1j}, & a_{2j}, & \ldots\ldots, & a_{nj} \\ b_{1j}, & b_{2j} & \ldots\ldots, & b_{nj} \end{pmatrix}, \qquad j = 1, 2, \ldots\ldots, n,$$

which do not make up a Latin system, but partaking with it of the property of containing among them all the n^2 substitution-element $a \to b$. P_i and Q_j contain just one substitution-element $a_{ij} \to b_{ij}$ in common, corresponding to the constituent (a_{ij}, b_{ij}) of the Eulerean square.

An Eulerean square may be conveniently represented by a set of n^2

quaternary symbols (i, j, k, l): where the ith. row and jth. column of the matrix meet, stands the constituent (k, l). Any two of these symbols differ in at least three of the corresponding terms. This character of representing an Eulerean square is not affected by a permutation of the four terms of the symbol; thus*

(i) (i, j, a, b), (ii) (i, b, j, a), (iii) (i, a, j, b),
(iv) (i, j, b, a), (v) (i, a, b, j), (vi) (i, b, a, j)

represnt respectively the Eulerean squares given in (1).

2. The so-called regular representation of finite groups give a multitude of Latin systems of substitutions

$$P_x = \begin{pmatrix} y \\ yx \end{pmatrix},$$

x being a fixed element and y a variable ranging over all the elements of a group G. Taking this as intermediate system, we try to construct an Eulerean square

$$(x, y, \varphi(x, y), \varphi(x, y)x).$$

For this it is nescessary that $\varphi(x, y)$ for a fixed x and $\varphi(x, y)$ as well as $\varphi(x, y)x$ for a fixed y should range over all the elements of G. Writing $\varphi(x)$ for $\varphi(x, y)$, $\varphi(x)x$ must range over all the elements of G. This condition is sufficient, since then

$$(x, y, \varphi(xy), \varphi(xy)x)$$

clearly represents an Eulerean square.

(1°) If $G = G_1 \times G_2$ is a direct product, then from the representations of the Eulerean squares corresponding to G_1 and G_2:

$$(x_1, y_1, \varphi_1(x_1y_1), \varphi_1(x_1y_1)x_1), \quad (x_2, y_2, \varphi_2(x_2y_2), \varphi_2(x_2y_2)x_2),$$

we get the Eulerean square for G:

$$(x_1x_2, y_1y_2, \varphi_1(x_1y_1)\varphi_2(x_2y_2), \quad \varphi_1(x_1y_1)\varphi_2(x_2y_2)x_1x_2).$$

(2°) For a group of odd order G, it suffices to put $\varphi(x) = x^a$, a and $a+1$ being both prime to the order of G, for example $a = 1$.

(3°) If the ordeı of G is semi-even: $n = 2m$, the corresponding Eulerean square is impossible. Assume if possible the existence of $\varphi(x)$. The group G has a subgroup H of index 2. In fact, an element s of order 2 corresponds in the regular representation to a substitution of the form $(12)(34)......(n-1, n)$, which is an odd subsitution, n/2 being odd. The element s of G corresponding to the even substitutions in the representation form then a subgroup H of index 2. Suppose now that just for c elements x of H, $\varphi(x) \in H$, and for the remaining $m-c$ elements x of H, $\varphi(x) \in Hs$, so that for $m-c$ and c elements

* The first term must be kept in its place, if the Eulerean squares should remain normal, since in that case the set must contain the symbols $(1, j, j, j)$.

x of H and Hs, $\varphi(x) \in H$ and $\in Hs$, respectively, giving rise to exactly 2ϵ elements $\varphi(x) \in H$. H being of odd order m, this is impossible.

That an Eulerean square of semi-even order is alltogether impossible is a conjecture awaiting confirmation.

(4°) For an Abelian group, direct product of a cyclic group of order 2^n and a group of odd order, $\varphi(x)$ does not exist.

Proof. First let $G = \{x\}$, x being of order 2^n. Then, if $\varphi(x)$ exist,

$$\overset{a}{\Pi}\ x^a = \overset{a}{\Pi}\ \varphi(x^a) = x^{2^{n-1}(2^{n-1}-1)} = x^{2^{n-1}} \overset{n}{\Pi}\ \varphi(x^a)\, x^a = 1, \text{ a contradiction.}$$

The same method applies to the general case, as the product of all the elements of a group of odd order is unity.

(5°) For Abelian groups other than of type given in (4°), $\varphi(x)$ exists.

(a) Groups of type (2, 2, ..., 2).

$$x = a_1^{e_2}\, a_2^{e_2} \ldots a_r,^{e_r} \qquad \varphi(x) = a_1^{e_r} a_2^{e_1} a_3^{e_2} \ldots a_r.^{e_{r-1}+e_r}$$

The same holds good also for the type $(2^a, 2^a, \ldots\ldots, 2^a)$.

(b) Groups of type $(2^a, 2)$. Take for example $\alpha = 8$. The method is general.

$$\begin{array}{r|llll|llll|llll}
x = 1,\ a,\ a^2,\ a^3 & a^4, & a^5, & a^6, & a^7 & b, & ab, & a^2b, & a^3b & a^4b, & a^5b, & a^6b, & a^7b, \\
\varphi(x) = 1,\ a,\ a^2,\ a^3 & a^5b, & a^6b, & a^7b, & b & ab, & a^2b, & a^3b, & a^4b & a^4, & a^5, & a^6, & a^7, \\
\varphi(x)\,x = 1,\ a^2, a^4,\ a^6 & ab, & a^3b, & a^5b, & a^7b & a, & a^3, & a^5, & a^7 & b, & a^2b, & a^4b, & a^6b.
\end{array}$$

(c) If G/H is of type (a) or (b), then $\varphi(x)$ can be extended from H to G in the following manner. Let for example G/H be of type (a) with $r = 2$:

$$\begin{array}{llllll}
H \ni & x, & xa_1 & xa_2, & xa_1a_2 \\
H \ni & \varphi(x) = x', & x'a_2, & x\,a_1a_2, & x\,a_1 \\
H \ni & \varphi(x)\,x = x,'' & x''a_1a_2, & x''a_2^2a_1, & x''a^2_1a_2
\end{array}$$

As $x''a^2_2$ $x''a^2_1$ range over H, when x'' does so, all is good. The same method applies to the other cases.

(d) Now to establish our proposition, we may confine ourselves in virtue of (1°) to the case, where the group is of order 2^n and of rank $r = 2$ or 3.

If $r = 2$, $G = \{a_1, a_2\}$ of type $(2^\alpha, 2^\beta)$ we put $H = \{a^2_1, a^2_2\}$ and applying (c) we descend to H. By repeating the process we come to (a) or (b).

If $r = 3$, $G = \{a_1, a_2, a_3\}$ of type $(2^\alpha, 2^\beta, 2^\gamma)$, $\alpha \geqq \beta > \gamma$, we put $H = \{a^2, b^2, c^2\}$. Repeated application of (c) with G/H of type (a) brings us to $r = 2$. If $\alpha = \beta > \gamma$ put $H = \{a_1, a^2_2, a^2_3\}$ and apply (c). But then H is of type $(2^\alpha, 2^{\beta-1}, 2^{\gamma-1})$ and we are in the former case. Lastly, if $\alpha = \beta = \gamma$, we come to H of type $(2^{\alpha-1}, 2^{\alpha-1}, 2^{\alpha-1})$. Here, however, (a) is directly applicable to G.

Remark. No instance is known of an Eulerean square formed from a Latin square other than that derived from regular representation of a group.

3. Euler, who thought the square of order 6 impossible, has constructed a *deficient* square, vacant in two places but otherwise satisfying all demands. This we can do for any semi-even order: $n = 2m$. We mention here only a result, giving an intermediate and a transverse System P, Q, where all the indces i, j, s, t, etc. are to be read *modulo* n.

$$P_i = \binom{t}{t+i}, \qquad 0 \leqq i \leqq n-1,\ i \neq m-1,\ n-1.$$

$$P_{m-1} = \begin{pmatrix} t_1, & t_2 \\ t_1+m-1, & t_2-1 \end{pmatrix}, \quad t_1 = 0, 2, 4, \ldots, m-1, m, m+2, \ldots, n-1,$$

$$P_{n-1} = \begin{pmatrix} t_1, & t_2 \\ t_1-1, & t_2+m-1 \end{pmatrix}, \quad t_2 = 1, 3, 5, \ldots, m-2, m+1, m+3, \ldots, n-2.$$

$$j \text{ even}, \ Q_j = \begin{pmatrix} j+s_1, & j+s_2 \\ j+2s_1, & j+2s_2-1 \end{pmatrix}, \qquad \begin{matrix} s_1 = 0, 1, \ldots, m-1, \\ s_2 = m, m+1, \ldots, n-1, \end{matrix}$$

$$j \text{ odd}, \ Q_j = \begin{pmatrix} j+s_1', & j+s_2', & j+n-1 \\ j+2s_1', & j+2s_2'+1, & j-2 \end{pmatrix}, \quad \begin{matrix} s_1' = 0, 1, \ldots, m-2, \\ s_2' = m-1, m, \ldots, n-2. \end{matrix}$$

P_{m-1} and Q_0 have two elements $(m-1 \rightarrow n-2)$, $(m \rightarrow n-1)$ and P_{n-1} and Q_1 two elements $(0 \rightarrow n-1)$, $(n-1 \rightarrow n-2)$ in common; in each case one of the common elements is to be left out, while P_{m-1} with Q_1 and P_{n-1} with Q_0 have no element in common, the places $(m-1, 1)$ and $(n-1, 0)$ of the matrix are therefore vacant.

Ex. $n = 10$,

00	11	22	33	44	55	66	77	88	99
12	23	34	45	56	67	78	89	90	01
24	35	46	57	68	79	80	91	02	13
36	47	58	69	70	81	92	03	14	25
48	—	71	10	93	32	04	65	26	87
61	50	83	72	05	94	27	16	49	38
73	62	95	84	17	06	39	28	51	40
85	74	07	96	29	18	41	30	63	52
97	86	19	08	31	20	53	42	75	64
—	98	60	21	82	43	15	54	37	76