67. On Mercer's Theorem.

By Gen-ichirô SUNOUCHI.

Mathematical Institute, Tohoku Imperial University, Sendai.

(Comm. by. S. KAKEYA, M. I. A., Dec. 12, 1946.)

L. C Bosanquet [1] and G. Hayashi [2] proved the absolute convergence analogue of Mercer's limit theorem. Mercer's theorem is generalized by Copson-Ferrar, S. Izumi, J. Karamata and others. In this paper we prove the absolute convergence analogue of the generalized form of Mercer's theorem. The formulation and the proof runs parallel as the author s former paper [3]

Theorem 1. *If* (1) $a_n > 0$, (2) $\sum_{n=1}^{\infty} 1/a_n = \infty$ *and*

(3) $$y_n = (1+a_n)t_n - a_n t_{n-1} \quad (n = 1, 2, \ldots\ldots),$$

then $\sum_{n=1}^{\infty} |\Delta y_n| < \infty$ implies $\sum_{n=1}^{\infty} |\Delta t_n| < \infty$.

Proof. Putting $t_0 = 0$ and $\alpha_n = 1/a_n$ and solving (3) concerning t_n, we get

$$t_n = \frac{\alpha_1}{\prod_{\nu=1}^{n}(1+\alpha\ )} y_1 + \frac{\alpha_2(1+\alpha_1)}{\prod_{\nu=1}^{n}(1+\alpha_\nu)} y_2 + \ldots + \frac{\alpha_n(1+\alpha_1)\ldots(1+\alpha_{n-1})}{\prod_{\nu=1}^{n}(1+\alpha_\nu)} y_n,$$

and then

$$\Delta t_n = \frac{\alpha_n}{\prod_{\nu=1}^{n}(1+\alpha_\nu)} \{ -\alpha_1 y_1 - \alpha_2(1+\alpha_1)y_2 - \alpha_3(1+\alpha_1)(1+\alpha_2)y_3 - \ldots$$

$$- \alpha_{n-1}(1+\alpha_1)\ldots(1+\alpha_{n-2})y_{n-1} + (1+\alpha_1)\ldots(1+\alpha_{n-1})y_n \}$$

$$= \frac{\alpha_n}{\prod_{\nu=1}^{n}(1+\alpha_\nu)} \sum_{\mu=1}^{n} [\{\prod_{\nu=1}^{\mu-1}(1+\alpha_\nu)\} (\Delta y_\mu)]$$

where $y_0 = 1$ and $\prod_{\nu=1}^{0}(1+\alpha_\nu) = 1$. Thus

$$\sum_{n=1}^{\infty} |\Delta t_n| = \sum_{n=1}^{\infty} \left| \frac{\alpha_n}{\prod_{\nu=1}^{n}(1+\alpha_\nu)} \sum_{\mu=1}^{n} [\{\prod_{\nu=1}^{\mu-1}(1+\alpha_\nu)\} (\Delta y_\mu)] \right|$$

---

[1] Bosanquet, L. C., *An analogue of Mercer's theorem*, Journ. London Math. Soc., **13** (1938), 177-180.

[2] Hayashi, G., *A theorem on limit*, Tohoku Math. Journ., **45** (1939), 329-331.

[3] Sunouchi, G., *Theorems on limits of recurrent sequences*, Proc. Imperial Acad. Tokyo, **10** (1934), 4-7.

$$\leqq \sum_{\nu=1}^{\infty} \frac{a_n}{\prod_{\nu=1}^{n}(1+a_\nu)} \sum_{\mu=1}^{n} \left[\left\{\prod_{\nu=1}^{\mu-1}(1+a_\nu)\right\} \left| \Delta y_\mu \right| \right]$$

$$= \sum_{\mu=1}^{\infty} \left| \Delta y_\mu \right| \prod_{\nu=1}^{\mu-1}(1+a_\nu) \sum_{n=\mu}^{\infty} \frac{a_n}{\prod_{\nu=1}^{n}(1+a_\nu)},$$

where

$$\sum_{n=\mu}^{\lambda} \frac{a_n}{\prod_{\nu=1}^{n}(1+a_\nu)} = \frac{1}{\prod_{\nu=1}^{\mu-1}(1+a_\nu)} - \frac{1}{\prod_{\nu=1}^{\lambda}(1+a_\nu)} \to \frac{1}{\prod_{\nu=1}^{\mu-1}(1+a_\nu)}, \text{ as } \lambda \to \infty$$

by (2). Therefore

$$\sum_{n=1}^{\infty} \left| \Delta t_n \right| \leqq \sum_{\mu=1}^{\infty} \left| \Delta y_\mu \right|.$$

Theorem 2. *If* $q > -1$, $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n / (\lambda_1 + \lambda_2 + \dots + \lambda_{n-1}) = \infty$, *and*

$$y_n = x_n + q \frac{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n}{\lambda_1 + \lambda_2 + \dots\dots + \lambda_n},$$

then $\sum_{n=1}^{\infty} \left| \Delta x_n \right| < \infty$ implies $\sum_{n=1}^{\infty} \left| \Delta y_n \right| < \infty$.

Proof. If we put

$$\Lambda_n = \lambda_1 + \lambda_2 + \dots\dots + \lambda_n,$$
$$t_n = (\lambda_1 x_1 + \dots\dots\dots + \lambda_n x_n) / \Lambda_n,$$

then

$$y_n = \left(1 + q + \frac{\Lambda_{n-1}}{\lambda_n}\right) t_n - \frac{\Lambda_{n-1}}{\lambda_n} t_{n-1}.$$

Dividing both sides by $1+q$,

$$\frac{y_n}{1+q} = \left(1 + \frac{\Lambda_{n-1}}{(1+q)\lambda_n}\right) t_n - \frac{\Lambda_{n-1}}{(1+q)\lambda_n} t_{n-1}.$$

Putting $a_n = \Lambda_{n-1} / (1+q)\lambda_n$ and applying Theorem 1, we get Theorem 2.

Theorem 2 contains theorems due to Bosanquet and Hayashi.