57. On the Infinitesimal Deformations of Curves in the Spaces with Linear Connection.

By Kentaro YANO, Kazuo TAKANO, and Yasuro TOMONAGA.

Mathematical Institute, Tokyo Imperial University.

(Comm. by S. KAKEYA, M. I. A., Nov. 12, 1946.)

§ 0. *Introduction.*

Since T. Levi-Civita[1] has published his famous paper on the geodesic deviation, the theory of infinitesimal deformations of the curves was studied by T. Boggio, E. Bortolotti, E. Cartan, U. Crudeli, E. T. Davies, L. P. Eisenhart, A. De Mira Fernandes, H. A. Hayden, V. Hlavaty, M. S. Knebelman, A. J. McConnell, O. Onicescu, J. A. Schouten, J. L. Synge, G. Vranceanu and others. The theory of infinitesimal deformations of curves was then generalized to that of subspaces by E. Bortolotti, E. Cartan, E. T. Davies, H. A. Hayden, A. J. McConnell, J. A. Schouten, A. G. Walker, C. G. Weatherburn, K. Yano and others. Recently, the theory of infinitesimal deformations of the space itself was studied by N. Coburn, D. v. Dantzig, E. T. Davies, P. Dienes, L. P. Eisenhart, E. R. van Kampen, M. S. Knebelman, A. J. McConnell, J. A. Schouten, W. Slebodzinski, K. Yano and others.

One of the present authors[2] has recently developed a geometrical theory of infinitesimal deformations and studied especially the deformations of subspaces imbedded in a space with linear connection.

In the present paper, we shall apply these methods, those of the above mentionned authors and that of K. Yano, to the study of the infinitesimal deformations of curves, particularly, of geodesics, affine conics, projective conics, geodesic circles and conformal circles. We shall state here only the results, the full detail will be published elsewhere.

§ 1. *The definition of the operator Δ.*

Let us consider a space of n dimensions with an affine connection $\Gamma^{\lambda}_{\mu\nu}$. A curve in the space being represented by the equations of the form $x^\lambda = x^\lambda(t)$, where t is an arbitrary parameter, we shall consider the infinitesimal deformation

$$\overline{x}^\lambda(\overline{t}) = x^\lambda(t) + \xi^\lambda(t)\,\delta u. \tag{1.1}$$

---

(1) T. Levi-Civita: Sur l'écart géodésique. Math. Ann., *97* (1926), 291–320.

(2) K. Yano: Sur la théorie des déformations infinitésimales. To appear in the Journal of the Faculty of Science, Imperial University of Tokyo.

of the curve, where t is the parameter of the deformed curve and $\xi^\lambda(t)$ is a contravariant vector field defined along the original curve, δu being an infinitesimal constant.

If we consider a contravariant vector field $v^\lambda(t)$ defined with respect to the curve $x^\lambda(t)$ and the parameter t——for example, the vector field $v^\lambda(t)$ may be the tangent vector $\frac{dx^\lambda}{dt}$, —— we have, after the infinitesimal deformation (1.1), a contravariant vector field $\bar{v}^\lambda(\bar{t})$ defined with respect to the deformed curve $\bar{x}^\lambda(\bar{t})$ and the new parameter $\bar{t}$. The vectors $v^\lambda(t)$ and $\bar{v}^\lambda(\bar{t})$ being defined at the point $x^\lambda(t)$ and $\bar{x}^\lambda(\bar{t})$ respectively, we transport parallelly the vector $v^\lambda(t)$ from the point $x^\lambda(t)$ to the infinitesimally near point $x^\lambda(\bar{t})$, obtaining

$$*v^\lambda(t) = v^\lambda(t) - \Gamma^\lambda_{\mu\nu} v^\mu \xi^\nu \delta u$$

at the point $\bar{x}^\lambda(\bar{t})$. Putting

$$\Delta v^\lambda = \bar{v}^\lambda(\bar{t}) - *v^\lambda(t) = \bar{v}^\lambda(\bar{t}) - v^\lambda(t) + \Gamma^\lambda_{\mu\nu} v^\mu \xi^\nu \delta u, \tag{1.2}$$

we shall call it the Δ-variation of the vector v^λ with respect to ξ^λ.(3)

For the Δ-variation of the parameter, we put

$$\Delta\, dt = d\bar{t} - dt, \tag{1.3}$$

from which we obtain

$$\frac{d\bar{t}}{dt} = 1 + \frac{\Delta\, dt}{dt} \quad \text{and} \quad \frac{dt}{d\bar{t}} = 1 - \frac{\Delta\, dt}{dt}. \tag{1.4}$$

Strictly speaking, the vectors $\bar{v}^\lambda(\bar{t})$ and $*v^\lambda(t)$ are defined at the point $\bar{x}^\lambda(\bar{t})$ and not at the original point $x^\lambda(\bar{t})$, and consequently the Δ-variation Δv^λ is a vector at the point $\bar{x}^\lambda(\bar{t})$. To obtain the vectors at the original point $x^\lambda(t)$, we transport parallelly these vectors from $\bar{x}^\lambda(t)$ back to $x^\lambda(t)$, obtaining

$$*(\Delta v^\lambda) = *\bar{v}^\lambda(\bar{t}) - v^\lambda(t).$$

But the vectors $*(\Delta v^\lambda)$ and Δv^λ coincide in the order of approximation of the first order, thus we have

$$*\bar{v}^\lambda(\bar{t}) = v^\lambda(t) + \Delta v^\lambda \tag{1.5}$$

at the original point.

If we have a tensor field defined at every point of the space, its Δ-variation will be the covariant derivative in the direction ξ^λ multiplied by the infinitesimal constant δu.

(3) H. A. Hayden: Deformations of a curve, in a Riemannian n-space, which displace certain vectors parallelly at each point. Proc. London Math. Soc., *32* (1931), 321–336.

J. A. Schouten and E. R. van Kampen: Beiträge zur Theorie der Deformation. Prace Mat. Fiz., 41 (1933), 1–19.

§ 2. *The definition of the operator D.*

In this Paragraph, we shall consider an infinitesimal deformation

(2.1) $$\bar{x}^\lambda = x^\lambda + \xi^\lambda(x)\,\delta u,$$

which displaces every point of the space to an infinitesimally near point of the space.

If we have a curve $x^\lambda = x^\lambda(t)$ in the space, then, this curve will be, by the infinitesimal deformation (2.1), transformed into the curve

(2.2) $$\bar{x}^\lambda(\bar{t}) = x^\lambda(t) + \xi^\lambda(x(t))\,\delta u.$$

Consider a vector $v^\lambda(t)$ defined with respect to the original curve and the parameter t —— for example the covariant derivative $\frac{\delta^2 x^\lambda}{dt^2}$ of the tangent vector $\frac{dx^\lambda}{dt}$. The point $x^\lambda(t)$ on the curve being transported to the point $x^\lambda(t)$ given by (2.2), the point $x^\lambda(t) + v^\lambda(t)\,\delta v$ on the vector will be transported to the point

$$x^\lambda + v^\lambda(t)\,\delta v + \xi^\lambda(x+v(t)\,\delta v)\,\delta u = x^\lambda + v^\lambda(t)\,\delta v + \xi^\lambda(x)\,\delta u + \xi^\lambda_{,\nu}\, v^\nu\,\delta u\,\delta v,$$

where δv is an infinitesimal constant and the comma denotes the ordinary differentiation. Thus, we can consider that the vector $v^\lambda = [(x^\lambda + v^\lambda\,\delta v) - x^\lambda] \div \delta v$ is dragged from the point x^λ to the nearby point $\bar{x}^\lambda$ and gives

$$'v^\lambda(t) = \{[x^\lambda + v^\lambda(t)\,\delta v + \xi^\lambda(x+v(t)\,\delta v)\,\delta u] - [x^\lambda + \xi^\lambda\,\delta u]\} / \delta v$$
$$= v^\lambda(t) + \xi^\lambda_{,\nu} v^\nu(t)\,\delta u.$$

Thus, we obtain two vectors $v^\lambda(t)$ and $'v^\lambda(t)$ at the point x^λ. Putting

(2.3) $$Dv^\lambda = v^\lambda(t) - v^\lambda(t) = v^\lambda(\bar{t}) - v^\lambda(t) - \xi^\lambda_{,\nu} v^\nu\,\delta u,$$

we shall call it the D-variation of the vector v^λ with respect to ξ^λ.(4)

For the D-variation of the parameter t, we put

(2.4) $$D\,dt = dt - dt,$$

from which we obtain

(2.5) $$\frac{d\bar{t}}{dt} = 1 + \frac{D\,dt}{dt} \quad \text{and} \quad \frac{dt}{d\bar{t}} = 1 - \frac{D\,dt}{dt}.$$

The vectors $v^\lambda(\bar{t})$ and $'v^\lambda(t)$ and consequently $D\,v^\lambda$ being the vectors at the deformed point $\bar{x}^\lambda$, to obtain the vectors at the original point x^λ, we have only to dragg back these vectors from the point $\bar{x}^\lambda$ to the point x^λ, obtaining

(2.6) $$'(D\,v^\lambda) = '\bar{v}^\lambda(\bar{t}) - v^\lambda(t),$$

or

(2.7) $$'\bar{v}^\lambda(\bar{t}) = v^\lambda(t) + Dv^\lambda$$

to the order of our approximation.

---

(4) K. Yano, loc. cit. See (2).

If we have a tensor field, for exmple $T^{\lambda}_{\cdot\mu\nu}$, the D-variation of the tensor will be given by the formulae.

(2.8) $DT^{\lambda}_{\cdot\mu\nu}=[T^{\lambda}_{\cdot\mu\nu,\omega}\xi^{\omega}-T^{\alpha}_{\cdot\mu\nu}\xi^{\lambda}_{,\alpha}+T^{\lambda}_{\cdot\alpha\nu}\xi^{\alpha}_{,\mu}+T^{\lambda}_{\cdot\mu\alpha}\xi^{\alpha}_{,\nu}]\,\delta u,$

or, in the tensor form,

(2.9) $DT^{\lambda}_{\cdot\mu\nu}=[T^{\lambda}_{\cdot\mu\nu;\omega}\xi^{\omega}-T^{\alpha}_{\cdot\mu\nu}\xi^{\lambda}_{\cdot\alpha}+T^{\lambda}_{\cdot\alpha\nu}\xi^{\alpha}_{\cdot\mu}+T^{\lambda}_{\cdot\mu\alpha}\xi^{\alpha}_{\cdot\nu}]\,\delta u,$

where

(2.10) $$\xi^{\lambda}_{\cdot\alpha}=\xi^{\lambda}_{;\alpha}+S^{\lambda}_{\cdot\alpha\beta}\xi^{\beta},$$

and the semi-colon denotes the covariant differentiation with respect to the affine connection $\Gamma^{\lambda}_{\mu\nu}$.

The operation D is sometimes called the Lie derivative. It will be applied also to an affine connection $\Gamma^{\lambda}_{\mu\nu}$, giving

(2.11) $D\Gamma^{\lambda}_{\mu\nu}=[\xi^{\lambda}_{,\mu,\nu}+\Gamma^{\lambda}_{\mu\nu,\omega}\xi^{\omega}-\Gamma^{\alpha}_{\mu\nu}\xi^{\lambda}_{,\alpha}+\Gamma^{\lambda}_{\alpha\nu}\xi^{\alpha}_{,\mu}+\Gamma^{\lambda}_{\mu\alpha}\xi^{\alpha}_{,\nu}]\,\delta u,$

or, in tensor form,

(2.12) $$D\Gamma^{\lambda}_{\mu\nu}=[\xi^{\lambda}_{\cdot\mu;\nu}+R^{\lambda}_{\cdot\mu\nu\omega}\xi^{\omega}]\,\delta u.$$ [5]

§ 3. *The fundamental formulae for the operator Δ.*

We shall consider, in this Paragraph, the commutator operator $\Delta\frac{\delta}{dt}-\frac{\delta}{dt}\Delta$ of the operator Δ and the operator $\frac{\delta}{dt}$, $\frac{\delta}{dt}$ denoting the covariant differentiation along the curve, that is to say,

(3.1) $$\frac{\delta}{dt}v^{\lambda}=\frac{dv^{\lambda}}{dt}+\Gamma^{\lambda}_{\mu\nu}v^{\mu}\frac{dx^{\nu}}{dt}.$$

After some calculation, we have[6]

(3.2) $$\Delta\frac{\delta}{dt}v^{\lambda}-\frac{\delta}{dt}\Delta v^{\lambda}=R^{\lambda}_{\cdot\mu\nu\omega}v^{\mu}\frac{dx^{\nu}}{dt}\xi^{\omega}\,\delta u-\frac{\delta v^{\lambda}}{dt}\frac{\Delta\,dt}{dt},$$

where $R^{\lambda}_{\cdot\mu\nu\omega}$ denotes the curvature tensor formed with $\Gamma^{\lambda}_{\mu\nu}$. These formulae will be fundamental for the following discussion.

§ 4. *The fundamental formulae for the operator D.*

If we calculate the similar expression $D\frac{\delta}{dt}v^{\lambda}-\frac{\delta}{dt}Dv^{\lambda}$ as above, we obtain[7]

(4.1) $$D\frac{\delta}{dt}v^{\lambda}-\frac{\delta}{dt}Dv^{\lambda}=(D\Gamma^{\lambda}_{\mu\nu})v^{\mu}\frac{dx^{\nu}}{dt}-\frac{\delta v^{\lambda}}{dt}\frac{D\,dt}{dt},$$

which will be also fundamental for the following discussion.

§ 5. *The Δ-variations of the successive covariant derivatives of the tangent vector.*

Let us consider a curve $x^{\lambda}=x^{\lambda}(t)$ and its infinitesimal deformation (1.1).

(5) K. Yano: Bemerkungen zur infinitesimalen Deformationen eines Raumes. Proc. Imp. Acad. Tokyo, 21 (1945), 171–178.

(6) H. A. Hayden: loc. cit. See (3).

(7) E. T. Davies: On the infinitesimal deformation of a space. Annali di Mat., 12 (1933–1934), 145–151.

Then, applying the Δ-operator to the tangent vector $\frac{dx^\lambda}{dt}$, we obtain

$$(5.1)\qquad \Delta\frac{dx^\lambda}{dt}=\left(\frac{\delta\xi^\lambda}{dt}+S^{\lambda}_{\cdot\mu\nu}\frac{dx^\mu}{dt}\xi^\nu\right)\delta u-\frac{dx^\lambda}{dt}\frac{\Delta\, dt}{dt}.$$

The Δ-variation of the tangent vector being thus obtained, to obtain the Δ-variation of the covariant derivative of the tangent vector, we put $v^\lambda=\frac{dx^\lambda}{dt}$ in (3.2) and substitute (5.1), then we obtain

$$(5.2)\quad \Delta\frac{\delta^2x^\lambda}{dt^2}=\left[\frac{\delta}{dt}\left(\frac{\delta\xi^\lambda}{dt}+S^{\lambda}_{\cdot\mu\nu}\frac{dx^\mu}{dt}\xi^\nu\right)+R^{\lambda}_{\cdot\mu\nu\omega}\frac{\delta x^\mu}{dt}\frac{dx^\nu}{dt}\xi^\omega\right]\delta u$$
$$-2\frac{\delta^2x^\lambda}{dt^2}\frac{\Delta\, dt}{dt}-\frac{dx^\lambda}{dt}\frac{d}{dt}\frac{\Delta\, dt}{dt}.$$

The Δ-variation of the acceleration vector being thus obtained, to obtain the Δ-variation of the second covariant derivative of the tangent vector, we put $v^\lambda=\frac{\delta^2x^\lambda}{dt^2}$ in (3.2) and substitute (5.2), then we find

$$(5.3)\quad \Delta\frac{\delta^3x^\lambda}{dt^3}=\left[\frac{\delta^2}{dt^2}\left(\frac{\delta\xi^\lambda}{dt}+S^{\lambda}_{\cdot\mu\nu}\frac{dx^\mu}{dt}\xi^\nu\right)+\frac{\delta}{\delta t}\left(R^{\lambda}_{\cdot\mu\nu\omega}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\xi^\omega\right)\right.$$
$$\left.+R^{\lambda}_{\cdot\mu\nu\omega}\frac{\delta^2x^\mu}{dt^2}\frac{dx^\lambda}{dt}\xi^\omega\right]\delta u-3\frac{\delta^3x^\lambda}{dt^3}\frac{\Delta\, dt}{dt}-3\frac{\delta^2x^\lambda}{dt^2}\frac{d}{dt}\frac{\Delta\, dt}{dt}$$
$$-\frac{dx^\lambda}{dt}\frac{d^2}{dt^2}\frac{\Delta\, dt}{dt}.$$

Continuing in this way, we can find the Δ-variation of the successive covariant derivatives of the tangent vector by the recurrence formulae

$$(5.4)\quad \Delta\frac{\delta^a x^\lambda}{dt^a}=\frac{\delta}{dt}\Delta\frac{d^{a-1}x^\lambda}{dt^{a-1}}+R^{\lambda}_{\cdot\mu\nu\omega}\frac{\delta^{a-1}x^\mu}{dt^{a-1}}\frac{dx^\nu}{dt}\xi^\omega\delta u-\frac{\delta^a x^\lambda}{dt^a}\frac{\Delta\, dt}{dt}.$$

§6. *The D-variations of the successive covariant derivatives of the tangent vector.*

The calculation analogous to that of the preceding Paragraph will give the D-variations of the successive covariant derivatives of the tangent vector as follows:

$$(6.1)\quad D\frac{dx^\lambda}{dt}=-\frac{dx^\lambda}{dt}\frac{D\,dt}{dt},$$

$$(6.2)\quad D\frac{\delta^2x^\lambda}{dt^2}=(D\,\Gamma^\lambda_{\mu\nu})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}-2\frac{\delta^2x^\lambda}{dt^2}\frac{D\,dt}{dt}-\frac{dx^\lambda}{dt}\frac{d}{dt}\frac{D\,dt}{dt},$$

$$(6.3)\quad D\frac{\delta^3x^\lambda}{dt^3}=(D\Gamma^\lambda_{\mu\nu})_{;\omega}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{dx^\omega}{dt}+2(D\Gamma^\lambda_{\mu\nu})\frac{\delta^2x^\mu}{dt^2}\frac{dx^\nu}{dt}+(D\Gamma^\lambda_{\mu\nu})\frac{dx^\mu}{dt}\frac{\delta^2x^\nu}{dt^2}$$
$$-3\frac{\delta^3x^\lambda}{dt^3}\frac{Ddt}{dt}-3\frac{\delta^2x^\lambda}{dt^2}\frac{d}{dt}\frac{Ddt}{dt}-\frac{dx^\lambda}{dt}\frac{d^2}{dt^2}\frac{D\,dt}{dt},$$

the recurrence formulae being

$$(6.4)\quad D\frac{\delta^a x^\lambda}{dt^a}=\frac{\delta}{dt}D\frac{\delta^{a-1}x^\lambda}{dt^{a-1}}+(D\,\Gamma^\lambda_{\mu\nu})\frac{\delta^{a-1}x^\mu}{dt^{a-1}}\frac{dx^\nu}{dt}-\frac{\delta^a x^\lambda}{dt^a}\frac{D\,dt}{dt}.$$

§7. *The Δ-variations of the normals and curvatures in the deformation of a curve in a Riemannian space.*

Let us consider an n-dimensional Riemannian space V_n whose fundamental form is $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ and a curve $x^\lambda = x^\lambda(s)$ in the space, s being the arc length of the curve. Then, the formulae of Frenet and Serret may be written as

$$(7.1) \qquad \frac{\delta}{ds}\eta^\lambda_{(a)} = -\kappa_{(a-1)}\,\eta^\lambda_{(a-1)} + \kappa_{(a)}\,\eta^\lambda_{(a+1)},$$

$$(a = 1, 2, \ldots, n;\ \kappa_{(o)} = \kappa_{(n)} = 0)$$

where $\eta^\lambda_{(1)}, \eta^\lambda_{(2)}, \ldots, \eta^\lambda_{(n)}$ are the unit tangent, the first normal, ..., the (n-1)-st normal of the curve and $\kappa_{(1)}, \kappa_{(2)}, \ldots, \kappa_{(n-1)}$ are the first, the second, ..., the (n-1)-st curvature of the curve respectively.

Now, we consider an infinitesimal deformation

$$(7.2) \qquad \bar{x}^\lambda(\bar{s}) = x^\lambda(s) + \xi^\lambda(s)\,\delta u$$

of the curve. The ds being transformed into $d\bar{s}$, if we put $\Delta\, ds = d\bar{s} - ds$, we find

$$(7.3) \qquad \frac{\Delta\, ds}{ds} = g_{\mu\nu}\frac{\delta\xi^\mu}{ds}\frac{dx^\nu}{ds}\,\delta u.$$

To obtain the Δ-variation of the unit tangent vector $\eta^\lambda_{(1)}$, we put $\dfrac{dx^\lambda}{dt} = \eta^\lambda_{(1)}$ and $t = s$ in (5.1), then we find

$$(7.4) \qquad \Delta\,\eta^\lambda_{(1)} = \frac{\delta\xi^\lambda}{ds}\,\delta u - \eta^\lambda_{(1)}\,g_{\mu\nu}\frac{\delta\xi^\mu}{ds}\frac{dx^\nu}{ds}\,\delta u.$$

To obtain the Δ-variation $\Delta\,\eta^\lambda_{(2)}$ of the first normal vector $\eta^\lambda_{(2)}$, we put $v^\lambda = \eta^\lambda_{(1)}$ and $t = s$ in (3.2) and substitute $\dfrac{\delta}{ds}\eta^\lambda_{(1)} = \kappa_{(1)}\eta^\lambda_{(2)}$, then we find

$$(7.5) \qquad \kappa_{(1)}\Delta\eta^\lambda_{(2)} = \frac{\delta}{ds}(\Delta\,\eta^\lambda_{(1)}) + R^\lambda_{\cdot\mu\nu\omega}\,\eta^\mu_{(1)}\eta^\nu_{(1)}\,\xi^\omega\,\delta u - \left[\Delta\kappa_{(1)} + \kappa_{(1)}\frac{\Delta\, ds}{ds}\right]\eta^\lambda_{(2)}.$$

Remembering $g_{\mu\nu}\,\eta^\mu_{(2)}\eta^\nu_{(2)} = 1$ and consequently $g_{\mu\nu}(\Delta\,\eta^\mu_{(2)})\,\eta^\nu_{(2)} = 0$ $(\Delta g_{\mu\nu} = 0)$, we have, from (7.5),

$$(7.6) \qquad \Delta\,\kappa_{(1)} = g_{\mu\nu}\frac{\delta}{ds}(\Delta\,\eta^\mu_{(1)})\,\eta^\nu_{(2)} + R_{\lambda\mu\nu\omega}\,\eta^\lambda_{(2)}\,\eta^\mu_{(1)}\,\eta^\nu_{(1)}\,\xi^\omega\,\delta u - \kappa_{(1)}\frac{\Delta\, ds}{ds},$$

which gives the Δ-variation $\Delta\,\kappa_{(1)}$ of the first curvature $\kappa_{(1)}$.

To obtain the Δ-variation $\Delta\,\eta^\lambda_{(3)}$ of the second normal vector $\eta^\lambda_{(3)}$, we put $v^\lambda = \eta^\lambda_{(2)}$ and $t = s$ in (3.2), and substitute $\dfrac{\delta}{ds}\eta^\lambda_{(2)} = -\kappa_{(1)}\,\eta^\lambda_{(1)} + \kappa_{(2)}\,\eta^\lambda_{(3)}$, then we obtain

$$(7.7) \qquad \kappa_{(2)}\,\Delta\eta^\lambda_{(3)} = \frac{\delta}{ds}(\Delta\,\eta^\lambda_{(2)}) + \kappa_{(1)}\,\Delta\eta^\lambda_{(1)} + R^\lambda_{\cdot\mu\nu\omega}\,\eta^\mu_{(2)}\,\eta^\nu_{(1)}\,\xi^\omega\,\delta u$$
$$- \left[\Delta\,\kappa_{(2)} + \kappa_{(2)}\frac{\Delta\, ds}{ds}\right]\eta^\lambda_{(3)} + \left[\Delta\kappa_{(1)} + \kappa_{(1)}\frac{\Delta\, ds}{ds}\right]\eta^\lambda_{(1)},$$

from which

$$(7.8) \qquad \Delta\,\kappa_{(2)} = g_{\mu\nu}\frac{\delta}{ds}(\Delta\eta^\mu_{(2)})\,\eta^\nu_{(3)} + \kappa_{(1)}\,g_{\mu\nu}(\Delta\eta^\mu_{(1)})\,\eta^\nu_{(3)} + R_{\lambda\mu\nu\omega}\,\eta^\lambda_{(3)}\,\eta^\mu_{(2)}\eta^\nu_{(1)}\,\xi^\omega\,\delta u$$
$$- \kappa_{(2)}\frac{\Delta\, ds}{ds},$$

by virtue of the identity $g_{\mu\nu}(\Delta\,\eta^\mu_{(3)})\,\eta^\nu_{(3)} = 0$.

Proceding in this manner, we can find the Δ-variation of the α-th normal $\eta^{\lambda}_{(a+1)}$:

(7.9) $$\kappa_{(a)}\Delta\eta^{\lambda}_{(a+1)} = \frac{\delta}{ds}\Delta\eta^{\lambda}_{(a)}) + \kappa_{(a-1)}\Delta\eta^{\lambda}_{(a-1)} + R^{\lambda}{}_{\cdot\mu\nu\omega}\eta^{\mu}_{(a)}\eta^{\nu}_{(1)}\xi^{\omega}\delta u - \left[\Delta\kappa_{(a)} + \kappa_{(a)}\frac{\Delta ds}{ds}\right]\eta^{\lambda}_{(a+1)} + \left[\Delta\kappa_{(a-1)} + \kappa_{(a-1)}\frac{\Delta ds}{as}\right]\eta^{\lambda}_{(a-1)},$$

and the Δ-variation of the α-th curvature:

(7.10) $$\Delta\kappa_{(a)} = g_{\mu\nu}\frac{\delta}{ds}(\Delta\eta^{\mu}_{(a)})\eta^{\nu}_{(a+1)} + \kappa_{(a-1)}g_{\mu\nu}(\Delta\eta^{\mu}_{(a-1)})\eta^{\nu}_{(a+1)} + R_{\lambda\mu\nu\omega}\eta^{\lambda}_{(a+1)} \times \eta^{\mu}_{(a)}\eta^{\nu}_{1)}\xi^{\omega}\delta u - \kappa_{(a)}\frac{\Delta ds}{ds}.$$

These formulae were found by A. J. McConnell.(8)

§8. *The D-variations of the normals and curvatures in the deformation of a curve in a Riemannian space.*

If we consider, in an n-dimensional Riemannian space, an infinitesimal deformation $\bar{x}^{\lambda} = x^{\lambda} + \xi^{\lambda}(x)\delta u$ of the space, a curve $x^{\lambda}(s)$ will be transformed into a curve $\bar{x}^{\lambda}(\bar{s}) = x^{\lambda}(s) + \xi^{\lambda}(x(s))\delta u$, and we can try the study quite analogous to that of the preceding Paragraph, replacing the Δ-variation by the D-variation, and we shall obtain the following results:

(8.1) $$\frac{Dds}{ds} = \frac{1}{2}(Dg_{\mu\nu})\eta^{\mu}_{(1)}\eta^{\nu}_{(1)} = \frac{1}{2}(\xi_{\mu;\nu} + \xi_{\nu;\mu})\eta^{\mu}_{(1)}\eta^{\nu}_{(1)}\delta u,$$

(8.2) $$D\eta^{\lambda}_{(1)} = -\eta^{\lambda}_{(1)}\frac{Dds}{ds},$$

(8.3) $$\kappa_{(1)}D\eta^{\lambda}_{(2)} = \frac{\delta}{ds}(D\eta^{\lambda}_{(1)}) + (D\Gamma^{\lambda}_{\mu\nu})\eta^{\mu}_{(1)}\eta^{\nu}_{(1)} - \left[D\kappa_{(1)} + \kappa_{(1)}\frac{Dds}{ds}\right]\eta^{\lambda}_{(2)},$$

(8.4) $$D\kappa_{(1)} = g_{\mu\nu}\frac{\delta}{ds}(D\eta^{\mu}_{(1)})\eta^{\nu}_{(2)} + (D\Gamma^{\lambda}_{\mu\nu})\eta_{(2)\lambda}\eta^{\mu}_{(1)}\eta^{\nu}_{(1)} - \kappa_{(1)}\frac{Dds}{ds} + \frac{1}{2}(Dg_{\mu\nu})\eta^{\mu}_{(2)}\eta^{\nu}_{(2)},$$

(8.5) $$\kappa_{(2)}D\eta^{\lambda}_{(3)} = \frac{\delta}{ds}(D\eta^{\lambda}_{(2)}) + \kappa_{(1)}D\eta^{\lambda}_{(1)} + (D\Gamma^{\lambda}_{\mu\nu})\eta^{\mu}_{(2)}\eta^{\nu}_{(1)} - \left[D\kappa_{(2)} + \kappa_{(2)}\frac{Dds}{ds}\right]\eta^{\lambda}_{(3)} + \left[D\kappa_{(1)} + \kappa_{(1)}\frac{Dds}{ds}\right]\eta^{\lambda}_{(1)},$$

(8.6) $$D\kappa_{(2)} = g_{\mu\nu}\frac{\delta}{ds}(D\eta^{\mu}_{(2)})\eta^{\nu}_{(3)} + (D\Gamma^{\lambda}_{\mu\nu})\eta_{(3)\lambda}\eta^{\mu}_{(2)}\eta^{\nu}_{(1)} - \kappa_{(2)}\frac{Dds}{ds} + \frac{1}{2}\kappa_{(1)}(Dg_{\mu\nu})\eta^{\mu}_{(3)}\eta^{\nu}_{(3)},$$

....

(8.7) $$\kappa_{(a)}D\eta^{\lambda}_{(a+1)} = \frac{\delta}{ds}(D\eta^{\lambda}_{(a)}) + \kappa_{(a-1)}D\eta^{\lambda}_{(a-1)} + (D\Gamma^{\lambda}_{\mu\nu})\eta^{\mu}_{(a)}\eta^{\nu}_{(1)} - \left[D\kappa_{(a)} + \kappa_{(a)}\frac{Dds}{ds}\right]\eta^{\lambda}_{(a+1)} + \left[D\kappa_{(a-1)} + \kappa_{(a-1)}\frac{Dds}{ds}\right]\eta^{\lambda}_{(a-1)},$$

(8.8) $$D\kappa_{(a)} = g_{\mu\nu}\frac{\delta}{ds}(D\eta^{\mu}_{(a)})\eta_{(a+1)} + \kappa_{(a-1)}g_{\mu\nu}(D\eta^{\mu}_{(a-1)}\eta^{\nu}_{(a+1)} + (D\Gamma^{\lambda}_{\mu\nu})\eta_{(a+1)\lambda}\eta^{\mu}_{(a)}\eta^{\nu}_{(1)}$$

(8) A. J. McConnell: The variation of curvatures in the deformation of a curve in Riemannian space. Proc. Irish Acad., *39* (1929–1930), 1–9.

$$-\kappa_{(a)}\frac{Dds}{ds}+\frac{1}{2}\kappa_{(a)}(Dg_{\mu\nu})\eta^{\mu}_{(a+1)}\eta^{\nu}_{(a+1)},$$
$$(a=1,2,\ldots,n-1).$$

§9. *Parallel tangent deformation.*

An infinitesimal deformation

$$\bar{x}^{\lambda}(\bar{t})=x^{\lambda}(t)+\xi^{\lambda}(t)\delta u \tag{9.1}$$

which displaces the curve $x^{\lambda}(t)$ into a curve $\bar{x}^{\lambda}(\bar{t})$ is said to be a parallel tangent deformation[9] when the tangent vector of the original curve is always parallel, in the sense of the affine connection, to the tangent vector of the deformed curve at the corresponding point. The necessary and sufficient condition that the infinitesimal deformation (9.1) be a parallel tangent deformation is that $\Delta\frac{dx^{\lambda}}{dt}=\alpha\frac{dx^{\lambda}}{dt}$, or

$$\left(\frac{\delta\xi^{\lambda}}{dt}+S^{\cdot\lambda}_{\cdot\mu\nu}\frac{dx^{\mu}}{dt}\xi^{\nu}\right)\delta u-\frac{dx^{\lambda}}{dt}\frac{\Delta dt}{dt}=\alpha\frac{dx^{\lambda}}{dt}. \tag{9.2}$$

If the ξ^{λ} is a contravariant vector field, then we have

$$(\xi^{\lambda}_{;\mu}+S^{\cdot\lambda}_{\cdot\mu\nu}\xi^{\nu})\frac{dx^{\mu}}{dt}=\beta\frac{dx^{\lambda}}{dt}, \tag{9.3}$$

Moreover, if the infinitesimal deformation $\bar{x}^{\lambda}=x^{\lambda}+\xi^{\lambda}(x)\delta u$ is always the parallel tangent deformation for any curve, then we have

$$\xi^{\lambda}_{;\mu}+S^{\cdot\lambda}_{\cdot\mu\nu}\xi^{\nu}=\varphi\delta^{\lambda}_{\mu}. \tag{9.4}$$

If we are in a Riemannian space, we have

$$\xi^{\lambda}_{;\mu}=\varphi\delta^{\lambda}_{\mu}\quad\text{or}\quad\xi_{\lambda;\mu}=\varphi g_{\lambda\mu}. \tag{9.5}$$

Thus, in order that there exist an infinitesimal deformation of the space which is a parallel tangent deformation for any curve in a Riemannian space, it is necessary and sufficient that the Riemannian space admit a concircular transformation.[10]

§10. *Combescure transformation.*

In a Riemannian space, if an infinitesimal deformation is such that

$$\Delta\eta^{\lambda}_{(1)}=\Delta\eta^{\lambda}_{(2)}=\ldots\ldots=\Delta\eta^{\lambda}_{(n)}, \tag{10.1}$$

it will be reffered as generalized Combescure transformation.[11]

For a generalized Combescure transformation, we have

$$\begin{cases}\dfrac{\delta\xi^{\lambda}}{ds}-\eta^{\lambda}_{(1)}g_{\mu\nu}\dfrac{\delta\xi^{\mu}}{ds}\eta^{\nu}_{(1)}=0,\\ R^{\lambda}_{\cdot\mu\nu\omega}\eta^{\mu}_{(a)}\eta^{\nu}_{(1)}\xi^{\omega}\delta u-\left[\Delta\kappa_{(a)}+\kappa_{(a)}\dfrac{\Delta ds}{ds}\right]\eta^{\lambda}_{(a+1)}\\ \qquad+\left[\Delta\kappa_{(a-1)}+\kappa_{(a-1)}\right]\dfrac{\Delta ds}{ds}\eta^{\lambda}_{(a-1)}=0,\end{cases} \tag{10.2}$$

(9) H. A. Hayden. loc. cit. See (3).

(10) K. Yano and T. Adati: Parallel tangent deformation, concircular transformation and concurrent vector field. Proc. Imp. Acad., Tokyo, **20** (1944), 123–127.

(11) H. A. Hayden, loc. cit. See (3).

and

$$(10.3)\qquad \varDelta\kappa_{(a)} + \kappa_{(a)}\frac{\varDelta\, ds}{ds} = R_{\lambda\mu\nu\omega}\eta^{\lambda}_{(a+1)}\,\eta^{\mu}_{(a)}\,\eta^{\nu}_{(1)}\,\xi^{\omega}\,\delta u.$$

Thus, if any curve in the space admits a Combescure transformation in any direction along the curve, we must have $R_{\lambda\mu\nu\omega}=0$, and consequently the space must be euclidean.(12) In this case the $\varDelta$-variation of the curvature is given by

$$(10.4)\qquad \varDelta\kappa_{(a)} + \kappa_{(a)}\frac{\varDelta\, ds}{ds} = 0.$$

§ 11. *Infinitesimal deformations which carry paths into paths.*

In this Paragraph, we shall consider the necessary and sufficient condition that the infinitesimal deformation carry a path into a path. Such a problem may be treated by the use of $\varDelta$-variation and of D-variation.

(i) *Method with the use of $\varDelta$-variation.*

By the definition of the $\varDelta$-variation, we have

$$(11.1)\qquad \varDelta\frac{\delta^2 x^\lambda}{ds^2} = \frac{\bar{\delta}^2\bar{x}^\lambda}{d\bar{s}^2} - {}^*\!\left(\frac{\delta^2 x^\lambda}{ds^2}\right).$$

The path being defined by $\frac{\delta^2 x^\lambda}{ds^2}=0$, we have the

Theorem 1: A necessary and sufficient condition that an infinitesimal deformation $\bar{x}^\lambda(\bar{s}) = x^\lambda(s) + \xi^\lambda(s)\,\delta u$ carry a path $x^\lambda(s)$ into a path $\bar{x}^\lambda(\bar{s})$ is

$$(11.2)\qquad \varDelta\frac{\delta^2 x^\lambda}{ds^2} = \left(\frac{\delta^2\xi^\lambda}{ds^2} + R^{\lambda}_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega\right)\delta u - \frac{dx^\lambda}{ds}\frac{d}{ds}\frac{\varDelta ds}{ds} = 0.^{(13)}$$

If we are in a Riemannian space, $\varDelta$-variation of the ds takes the form

$$\frac{\varDelta\, ds}{ds} = g_{\mu\nu}\frac{\delta\xi^\mu}{ds}\frac{dx^\nu}{ds}\delta u,$$

and hence the equations (11.2) become

$$(11.3)\qquad \frac{\delta^2\xi^\lambda}{ds^2} + R^{\lambda}_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega - \frac{dx^\lambda}{ds}g_{\mu\nu}\frac{\delta^2\xi^\mu}{ds^2}\frac{dx^\nu}{ds} = 0,$$

which are equations of "l'écart géodésique" of T. Levi-Civita.(14)

Returning to the affinely connected space, if the infinitesimal deformation which carries the path into path preserves the affine parameter, then we must have $\frac{\varDelta\, ds}{ds} = \varepsilon = \text{const.}$ Thus

Theorem 2: A necessary and sufficient condition that the infinitesimal deformation $x^\lambda(s) = x^\lambda(s) + \xi^\lambda(s)\,\delta u$ transform a path into path and preserve the affine parameter on the path is that

$$(114.)\qquad \frac{\delta^2\xi^\lambda}{ds^2} + R^{\lambda}_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega = 0.$$

(12) H. A. Hayden, loc. cit. See (3).

(13) The torsion tensor may be assumed to be zero tensor.

(14) T. Levi-Civita: loc. cit. See (1).

Now, if the ξ^λ is a contravariant vector field defined at each point of the space, the equations (11.3) and (11.4) become

$$(11.5)\qquad (\xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta u - \frac{dx^\lambda}{ds}\frac{d}{ds}\frac{\Delta ds}{ds} = 0$$

and

$$(11.6)\qquad (\xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

respectively. Thus we have the

Theorem 3: A necessary and sufficient condition that an infinitesimal deformation $\bar{x}^\lambda = x^\lambda + \xi^\lambda(x)\delta u$ *carry every path of the space into path is that*

$$(11.7)\qquad \xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega = \delta^\lambda_\mu\varphi_\nu + \delta^\lambda_\nu\varphi_\mu,$$

where φ_ν *is an arbitrary covariant vector.*

Such an infinitesimal transformation is called infinitesimal projective collineation[15] of the affinely connected space.

Theorem 4: A necessary and sufficient condition that an infinitesimal deformation of the space carry every path of the space into path and preserve the affine parameter on each path is that

$$(11.8)\qquad \xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega = 0.$$

Such an infinitesimal deformation is called infinitesimal affine collineation of the affinely connected space.

(*ii*) *Method with the use of D-variation.*

We consider an infinitesimal deformation $\bar{x}^\lambda = x^\lambda + \xi^\lambda(x)\delta u$ of the space. By the definition of the D-variation, we have

$$(11.9)\qquad D\frac{\delta^2 x^\lambda}{ds^2} = \frac{\bar{\delta}^2\bar{x}^\lambda}{d\bar{s}^2} - \left(\frac{\delta^2 x^\lambda}{ds^2}\right).$$

The path being defined by the equations $\frac{\delta^2 x^\lambda}{ds^2} = 0$, we have the

Theorem 5: A necessary and sufficient condition that an infinitesimal deformation $\bar{x}^\lambda = x^\lambda + \xi^\lambda(x)\delta u$ *carry a path into a path is*

$$(11.10)\qquad D\frac{\delta^2 x^\lambda}{ds^2} = (D\Gamma^\lambda_{\mu\nu})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{dx^\lambda}{ds}\frac{d}{ds}\frac{Dds}{ds}$$

$$= (\xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta u - \frac{dx^\lambda}{ds}\frac{d}{ds}\frac{Dds}{ds} = 0.$$

These equations coincide well with (11.5).

§ 12. *Infinitesimal deformations which carry affine conics into affine conics.*

An affine conic[16] in an affinely connected space is defined by the differential equations

(15) L. P. Eisenhart and M. S. Knebelman: Displacement in a geometry of paths which carry path into path. Proc. Nat. Acad. Sci., U. S. A., 13 (1927), 38–42.

(16) K. Yano and K. Takano, Sur les coniques dans les espaces à connexion affine ou projective, I, II. Proc. Imp. Acad. Tokyo, 20 (1944), 410–424.

$$(12.1)\qquad \frac{\delta^3 x^\lambda}{ds^3} + k\frac{dx^\lambda}{ds} = 0,$$

where k is a constant and s is an affine parameter. In this Paragraph, we shall consider the infinitesimal deformation which carry affne conics thus defined into affine conics of the space.

(i) *Method with the use of the Δ-variation.*

By the definition of the Δ-variation, we have

$$(12.2)\qquad \Delta\left(\frac{\delta^3 x^\lambda}{ds^3} + k\frac{dx^\lambda}{ds}\right) = \left(\frac{\bar{\delta}^3 \bar{x}^\lambda}{d\bar{s}^3} + \bar{k}\frac{d\bar{x}^\lambda}{d\bar{s}}\right) - {}^*\left(\frac{\delta^3 x^\lambda}{ds^3} + k\frac{dx^\lambda}{ds}\right).$$

Thus,

Theorem 6: A necessary and sufficient condition that an infinitesimal deformation carry an affine conic into an affine conic is that

$$(12.3)\qquad \Delta\left(\frac{\delta^3 x^\lambda}{ds^3} + k\frac{dx^\lambda}{ds}\right)$$

$$= \left[\frac{\delta^3\xi^\lambda}{ds^3} + k\frac{\delta\xi^\lambda}{ds} + \frac{\delta}{ds}\left(R^\lambda_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega\right) + R^\lambda_{\cdot\mu\nu\omega}\frac{\delta^2 x^\mu}{ds^2}\frac{dx^\nu}{ds}\xi^\omega\right]\delta u$$

$$-3\frac{\delta^2 x^\lambda}{ds^2}\frac{d}{ds}\frac{\Delta ds}{ds} + \frac{dx^\lambda}{ds}\left(\Delta k + 2k\frac{\Delta ds}{ds} - \frac{d^2}{ds^2}\frac{\Delta ds}{ds}\right) = 0.$$

Theorem 7: A necessary and sufficient condition that an infinitesimal deformation carry an affine conic into an affine conic and preserve the affine parameter on the affine conic is that

$$(12.4)\qquad \left[\frac{\delta^3\xi^\lambda}{ds^3} + k\frac{\delta\xi^\lambda}{ds} + \frac{\delta}{ds}\left(R^\lambda_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega\right) + R^\lambda_{\cdot\mu\nu\omega}\frac{\delta^2 x^\mu}{ds^2}\frac{dx^\nu}{ds}\xi^\omega\right]\delta u$$

$$+\frac{dx^\lambda}{ds}(\Delta k + 2k\varepsilon) = 0.$$

Now, we suppose that ξ^λ is a contravariant vector field in the space, then the equation (12 3) become

$$(12.5)\qquad (\xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)_{;\tau}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\tau}{ds} + 2(\xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)\frac{\delta^2 x^\mu}{ds^2}\frac{dx^\nu}{ds}$$

$$+(\xi^\lambda_{;\mu;\nu} + R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)\frac{dx^\mu}{ds}\frac{\delta^2 x^\nu}{ds} + \alpha\frac{\delta^2 x^\lambda}{ds^2} + \beta\frac{dx^\lambda}{ds} = 0,$$

and the equations (12.4) takes the same form with $\alpha = 0$.

From these equations we have the

Theorem 8: A necessary and sufficient condition that an infinitesimal transformation of the space carry an arbitrary affine conic of the space into an affine conic is that this infinitesimal transformation is a projective collineation.

Theorem 9:[17] *A necessary and sufficient condition that an infinitesimal transformation of the space carry an arbitrary affine conic of the space into an affine conic and preserve the affine parameter on the curve is that the infinite-*

(17) K. Yano and K. Takano: Quelques remarques sur les groupes de transformations dans les espaces à connexion linéaire, II. Proc. 22 (1946), 67–72.

simal transformation is an affine collineation.

(*ii*) *Method with the use of the D-variation.*

By the definition of the D-variation, we have the

Theorem 10: A necessary and sufficient condition that an infinitesimal deformation carry an affine conic into an affine conic is

$$(12.6)\quad D\left(\frac{\delta^3 x^\lambda}{ds^3}+k\frac{dx^\lambda}{ds}\right)$$
$$=(D\Gamma^\lambda_{\mu\nu})_{;\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\omega}{ds}+2(D\Gamma^\lambda_{\mu\nu})\frac{\delta^2x^\mu}{ds^2}\frac{dx^\nu}{ds}+(D\Gamma^\lambda_{\mu\nu})\frac{dx^\mu}{ds}\frac{\delta^2x^\nu}{ds^2}$$
$$-3\frac{\delta^2x^\lambda}{ds^2}\frac{d}{ds}\frac{Dds}{ds}-\frac{dx^\lambda}{ds}\left(\frac{d^2}{ds^2}\frac{Dds}{ds}-2k\frac{Dds}{ds}-Dk\right)=0\,.$$

The equation (12.6) coincides well with the equations (12.5).

§13. *Infinitesimal deformations which carry projective conics into projective conics.*

A projective conic and a projective parameter t on it are defined by the differential eqations of the form[18]

$$(13.1)\qquad \frac{d}{ds}(\{t,\ s\}+a^0)+\Gamma^0_{\mu\nu}\frac{\delta^2x^\mu}{ds^2}\frac{dx^\nu}{ds}=0,$$

$$(13.2)\qquad \frac{\delta^3x^\lambda}{ds^3}+[2\{t,\ s\}+a^0]\frac{dx^\lambda}{ds}=0,$$

where

$$\{t,\ s\}=\frac{\dfrac{d^3t}{ds^3}}{\dfrac{dt}{ds}}-\frac{3}{2}\left(\frac{\dfrac{d^2t}{ds^2}}{\dfrac{dt}{ds}}\right)^2,\quad a^0=\Gamma^0_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds},$$

$$\Gamma^0_{\mu\nu}=-\frac{1}{n^2-1}(n\,R_{\mu\nu}+R_{\mu\nu}),$$

$R_{\mu\nu}$ being the contracted curvature tensor. We shall consider the infinitesimal deformations of these projective conics.

(i) *Method with the use of the $\varDelta$-variation.*

By the definition of the $\varDelta$-variation, we have the

Theorem 11: A necessary and sufficient condition that an infinitesimal deformation carry a projective conic and preserve the projective parameter on them is that

$$(13.3)\quad \varDelta\left[\frac{d}{ds}(\{t,\ s\}+a^0)+\Gamma^0_{\mu\nu}\frac{\delta^2x^\mu}{ds}\frac{dx^\nu}{ds}\right]$$
$$=-(2\{t,\ s\}+3a^0)\frac{d}{ds}\frac{\varDelta ds}{ds}-\frac{d^3}{ds^3}\frac{\varDelta ds}{ds}$$
$$+\frac{d}{ds}\left(\Gamma^0_{\mu\nu;\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega+\Gamma^0_{\mu\nu}\frac{\delta\xi^\mu}{ds}\frac{dx^\nu}{ds}+\Gamma^0_{\mu\nu}\frac{dx^\mu}{ds}\frac{\delta\xi^\nu}{ds}\right)\delta u$$

(18) K. Yano and K. Takano: loc. cit. See (16).

$$+\left(\Gamma^0_{\mu\nu;\omega}\frac{\delta^2x^\mu}{ds^2}\frac{dx^\nu}{ds}\xi^\omega+\Gamma^0_{\mu\nu}\frac{\delta^2\xi^\mu}{ds^2}\frac{dx^\nu}{ds}+\Gamma^0_{\alpha\beta}R^\alpha_{\cdot\mu\nu\omega}\frac{dx^\beta}{ds}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega\right.$$
$$\left.+\Gamma^0_{\mu\nu}\frac{\delta^2x^\mu}{ds^2}\frac{\delta\xi^\nu}{ds}\right)\delta u=0$$

and

$$(13.4)\quad \Delta\left[\frac{\delta^3x^\lambda}{ds^3}+(2\{t,s\}+a^0)\frac{dx^\lambda}{ds}\right]$$
$$=\left[\frac{\delta^3\xi^\lambda}{ds^3}+(2\{t,s\}+a^0)\frac{\delta\xi^\lambda}{ds}+\frac{\delta}{ds}\left(R^\lambda_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega\right)\right.$$
$$\left.+R^\lambda_{\cdot\mu\nu\omega}\frac{\delta^2x^\mu}{ds^2}\frac{dx^\nu}{ds}\xi^\omega\right]\delta u+\left(\Gamma^0_{\mu\nu;\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega+\Gamma^0_{\mu\nu}\frac{\delta\xi^\mu}{ds}\frac{dx^\nu}{ds}\right.$$
$$\left.+\Gamma^0_{\mu\nu}\frac{dx^\mu}{ds}\frac{\delta\xi^\nu}{ds}\right)\frac{dx^\lambda}{ds}\delta u-3\frac{\delta^2x^\lambda}{ds^2}\frac{d}{ds}\frac{\Delta ds}{ds}-3\frac{dx^\lambda}{ds}\frac{d^2}{ds^2}\frac{\Delta ds}{ds}=0\,.$$

When ξ^λ are functions of x^λ, we have, from (13.4),

$$(13.5)\quad (\xi^\lambda_{;\mu;\nu;\omega}+R^\lambda_{\cdot\mu\nu\omega;\alpha}\xi^\omega+R^\lambda_{\cdot\mu\nu\alpha}\xi^\alpha_{;\omega})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\omega}{ds}\delta u$$
$$+2(\xi^\lambda_{;\mu;\nu}+R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)\frac{\delta^2x^\mu}{ds^2}\frac{dx^\nu}{ds}\delta u+(\xi^\lambda_{;\mu;\nu}+R^\lambda_{\cdot\mu\nu\omega}\xi^\omega)\frac{dx^\mu}{ds}\frac{\delta^2x^\nu}{ds^2}\delta u$$
$$+(\Gamma^0_{\mu\nu;\omega}\xi^\omega+\Gamma^0_{\alpha\nu}\xi^\alpha_{;\mu}+\Gamma^0_{\mu\alpha}\xi^\alpha_{;\nu})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}\delta u-3\frac{\delta^2x^\lambda}{ds^2}\frac{d}{ds}\frac{\Delta ds}{ds}$$
$$-3\frac{dx^\lambda}{ds}\frac{d^2}{ds^2}\frac{\Delta ds}{ds}=0.$$

If we assume that the infinitesimal deformation $\bar{x}^\lambda=x^\lambda+\xi^\lambda(x)\delta u$ of the space carries the projective conics into projective conics and preserve the projective parameters on them, we have, from (13.5),

$$(13.6)\qquad \xi^\lambda_{;\mu;\nu}+R^\lambda_{\cdot\mu\nu\omega}\xi^\omega=\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\varphi_\mu,$$

from which

$$(13.7)\qquad \Gamma^0_{\mu\nu;\omega}\xi^\omega+\Gamma^0_{\alpha\nu}\xi^\alpha_{;\mu}+\Gamma^0_{\mu\alpha}\xi^\alpha_{;\nu}=\varphi_{\mu;\nu}.$$

Conversely, if the vector ξ^λ satisfies the condition (13.6), it will be easily shown that the equations (13.3) and (13.4) are identically satisfied, and we have the

Theorem 12: A necessary and sufficient condition that an infinitesimal deformation of the space carry all the projective conics into projective conics and preserve the projective parameters on them is that the deformation is a projecive collineation.

(ii) *Method with the use of D-varia ion.*

By the definition of D-variation, we have the

Theorem 13: A necessary and sufficient condition that an infinitesimal deformation of the space $\bar{x}^\lambda=x^\lambda+\xi^\lambda(x)\delta u$ carry all the projective conics into projective conics and preserve the projective parameter on them is that we have

$$(13.8)\quad D\left[\frac{d}{ds}(\{t,s\}+a^0)+\Gamma^0_{\mu\nu}\frac{\delta^2x^\mu}{ds^2}\frac{dx^\nu}{ds}\right]$$

$$= -(2\{t, s\} + 3a^0)\frac{d}{ds}\frac{Dds}{ds} - \frac{d^3}{ds^3}\frac{Dds}{ds}$$

$$+\frac{d}{ds}\left[(D\Gamma^0_{\mu\nu})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \Gamma^0_{\mu\nu}\frac{\delta^2 x^\mu}{ds^2}\frac{dx^\nu}{ds}\frac{Dds}{ds} - \Gamma^0_{\mu\nu}\frac{dx^\mu}{ds}\frac{\delta^2 x^\nu}{ds^2}\frac{Dds}{ds}\right]\delta u$$

$$+(D\Gamma^0_{\mu\nu})\frac{\delta^2 x^\mu}{ds^2}\frac{dx^\nu}{ds} + (\Gamma^0_{\alpha\nu})(D\Gamma^\alpha_{\mu\omega})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\omega}{ds} = 0$$

and

(13.9) $$D\left[\frac{\delta^3 x^\lambda}{ds^3} + (2\{t, s\} + a^0)\frac{dx^\lambda}{ds}\right]$$

$$=(D\Gamma^\lambda_{\mu\nu});_\omega\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\omega}{ds} + 2(D\Gamma^\lambda_{\mu\nu})\frac{\delta^2 x^\mu}{ds^2}\frac{dx^\nu}{ds} + (D\Gamma^\lambda_{\mu\nu})\frac{dx^\mu}{ds}\frac{\delta^2 x^\nu}{ds^2}$$

$$-(D\Gamma^0_{\mu\nu})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} - 3\frac{\delta^2 x^\lambda}{ds^2}\frac{d}{ds}\frac{Dds}{ds} - 3\frac{dx^\lambda}{ds}\frac{d^2}{ds^2}\frac{Dds}{ds} = 0.$$

These equations coincide well with (13.5).

§ 14. *Infinitesimal deformations which carry Riemannian circles into Riemannian circles.*

Riemannian circles are defined by the differential equations of the form

(14.1) $$\frac{\delta^3 x^\lambda}{ds^3} + \kappa^2\frac{dx^\lambda}{ds} = 0, \quad \text{where} \quad \kappa^2 = g_{\mu\nu}\frac{\delta^2 x^\mu}{ds^2}\frac{\delta^2 x^\nu}{ds^2}.$$

We shall consider the infinitesimal deformations of these curves in Riemannian space.

By the definition of the Δ-variation, we have the

Theorem 14: A necessary and sufficient condition that an infinitesimal deformation $\bar{x}^\lambda(\bar{s}) = x^\lambda(s) + \xi^\lambda(s)\,\delta u$ carry a Riemannian circle into a Riemannian circle is that[19]

(14.2) $$\Delta\left(\frac{\delta^3 x^\lambda}{ds^3} + \kappa^2\frac{dx^\lambda}{ds}\right)$$

$$=\left[\frac{\delta^3\xi^\lambda}{ds^3} + \kappa^2\frac{\delta\xi^\lambda}{ds} + \frac{\delta}{ds}\left(R^\lambda_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega\right) + R^\lambda_{\cdot\mu\nu\omega}\frac{\delta^2 x^\mu}{ds^2}\frac{dx^\nu}{ds}\xi^\omega\right.$$

$$\left.-3\frac{\delta^2 x^\lambda}{ds^2}\left(g_{\mu\nu}\frac{\delta^2\xi^\mu}{ds^2}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{\delta\xi^\mu}{ds}\frac{\delta^2 x^\nu}{ds^2}\right)\right]\delta u$$

$$+\frac{dx^\lambda}{ds}\left(2\kappa\Delta\kappa + 3\kappa^2 g_{\mu\nu}\frac{\delta\xi^\mu}{ds}\frac{dx^\nu}{ds}\delta u - g_{\mu\nu}\frac{\delta^3\xi^\mu}{ds^3}\frac{dx^\nu}{ds}\delta u - 2g_{\mu\nu}\frac{\delta^2\xi^\mu}{ds^2}\frac{\delta^2 x^\nu}{ds^2}\delta u\right)$$

$$= 0,$$

the Δ-variation of κ being given by

(14.4) $$\kappa\Delta\kappa = \left[g_{\alpha\beta}\left(\frac{\delta^2\xi^\alpha}{ds^2} + R^\alpha_{\cdot\mu\nu\omega}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\omega\right)\frac{\delta^2 x^\beta}{ds^2} - 2\kappa^2 g_{\mu\nu}\frac{\delta^2\xi^\mu}{ds^2}\frac{dx^\nu}{ds}\right]\delta u.$$

If we assume that ξ^λ are functions of the coordinates, then we can apply either the method with the use of Δ-variation or the method with the use of D-variation and prove the following

(19) J. L. Synge: The displacement or deviation of circles in Riemannia space. Proc. Irish Acad., *39* (1929-1930), 10–20.

Theorem 15: A necessary and sufficient condition that an infinitesimal transformation of the space carry any Riemannian circle of the space into a Riemannian circle is that transformation is an infinitesimal concircular transformation,[20] *that is to say, the vector ξ^λ defining the transformation satisfy*

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2\phi\, g_{\mu\nu} \quad \text{and} \quad \phi_{\mu;\nu} = \rho g_{\mu\nu}. \tag{14.4}$$

§ 15. *Infinitesimal deformations which carry conformal circles into conformal circles.*

Conformal circles[21] in a Riemannian space and the projective parameter on them are defined by the differential equations of the form

$$\{t, s\} = \frac{1}{2} g_{\mu\nu} \frac{\delta^2 x^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds^2} - \Gamma^0_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}, \tag{15.1}$$

$$\frac{\delta^3 x^\lambda}{ds^3} + g_{\mu\nu} \frac{\delta^2 x^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds} \frac{dx^\lambda}{ds} - \Gamma^0_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} + \Gamma^\lambda_{\varkappa\nu} \frac{dx^\nu}{ds} = 0, \tag{15.2}$$

where

$$\Gamma^0_{\mu\nu} = -\frac{R_{\mu\nu}}{n-2} + \frac{g_{\mu\nu} R}{2(n-1)(n-2)}, \quad \Gamma^\lambda_{\varkappa\nu} = g^{\lambda a} \Gamma^0_{a\nu}.$$

We shall study the infitesimal deformations of these curves.

By the definition of the $\varDelta$-variation, we have the

Theorem 16: A necessary and sufficient condition that an infinitesimal deformation $\bar{x}^\lambda(\bar{s}) = x^\lambda(s) + \xi^\lambda(s)\,\delta u$ carry a conformal circle into a conformal circle and preserve the projective parameter on it is that

$$\begin{aligned}
&\varDelta\left(\{t, s\} - \frac{1}{2} g_{\mu\nu} \frac{\delta^2 x^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds^2} + \Gamma^0_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}\right) \\
&= \Big[- g_{\mu\nu} \frac{\delta^3 \xi^\mu}{ds^3} \frac{dx^\nu}{ds} - 2 g_{\mu\nu} \frac{\delta^2 \xi^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds^2} - \Gamma^0_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} g_{a\beta} \frac{\delta \xi^a}{ds} \frac{dx^\beta}{ds} \\
&\quad + 3\Gamma^0_{\mu\nu} \frac{\delta \xi^\mu}{ds} \frac{dx^\nu}{ds} - g_{a\beta}\left(\frac{\delta^2 \xi^a}{ds} + R^a_{\cdot\mu\nu\omega} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \xi^\omega\right) \frac{\delta^2 x^\beta}{ds^2} \\
&\quad + 2 g_{a\beta} \frac{\delta^2 x^a}{ds^2} \frac{\delta^2 x^\beta}{ds^2} g_{\mu\nu} \frac{\delta \xi^\mu}{ds} \frac{dx^\nu}{ds} + \Gamma^0_{\mu\nu;\omega} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \xi^\omega\Big]\delta u = 0
\end{aligned} \tag{15.3}$$

and

$$\begin{aligned}
&\varDelta\left[\frac{\delta^3 x^\lambda}{ds^3} + g_{\mu\nu} \frac{\delta^2 x^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds^2} \frac{dx^\lambda}{ds} - \Gamma^0_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} + \Gamma^\lambda_{\varkappa\nu} \frac{dx^\nu}{ds}\right] \\
&= \Big[\frac{\delta^3 \xi^\lambda}{ds^3} + g_{\mu\nu} \frac{\delta^2 x^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds^2} \frac{\delta \xi^\lambda}{ds} + \frac{\delta}{ds}\left(R^\lambda_{\cdot\mu\nu\omega} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \xi^\omega\right) \\
&\quad + R^\lambda_{\cdot\mu\nu\omega} \frac{\delta^2 x^\mu}{ds^2} \frac{dx^\nu}{ds} \xi^\omega - 3\frac{\delta^2 x^\lambda}{ds^2}\left(g_{\mu\nu} \frac{\delta^2 \xi^\mu}{ds^2} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{\delta \xi^\mu}{ds} \frac{\delta^2 x^\nu}{ds^2}\right) \\
&\quad + \frac{dx^\lambda}{ds}\left(2\kappa\varDelta\kappa + 3\kappa^2 g_{\mu\nu} \frac{\delta \xi^\mu}{ds} \frac{dx^\nu}{ds} - g_{\mu\nu} \frac{\delta^3 \xi^\mu}{ds^3} \frac{dx^\nu}{ds} - 2 g_{\mu\nu} \frac{\delta^2 \xi^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds^2}\right)
\end{aligned}$$

(20) K. Yano: Concircular geometry, I, II, III, IV, V. Proc. Imp. Acad., Tokyo, **16** (1940), 195-200, 354-360, 442-448, 505-511, **18**(1942), 446-451.

(21) K. Yano: Sur la théorie des espaces a connexion conforme. Journal of the Faculty of Science, Imperial University of Tokyo, **4**(1939), 1-59.

$$- \Gamma^0_{\mu\nu;\omega} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \xi^\omega \frac{dx^\lambda}{ds} - 2\,\Gamma^0_{\mu\nu} \frac{\delta\xi^\mu}{ds} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} + 2\,\Gamma^0_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$
$$\times g_{\alpha\beta} \frac{\delta\xi^\alpha}{ds} \frac{dx^\beta}{ds} \frac{dx^\lambda}{ds} - \Gamma^0_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \left(\frac{\delta\xi^\lambda}{ds} - \frac{dx^\lambda}{ds} g_{\alpha\beta} \frac{\delta\xi^\alpha}{ds} \frac{dx^\beta}{ds} \right)$$
$$+ \Gamma^\lambda_{0\nu;\omega} \frac{dx^\nu}{ds} \xi^\omega + \Gamma^\lambda_{0\nu} \frac{\delta\xi^\nu}{ds} - \Gamma^\lambda_{0\nu} \frac{dx^\nu}{ds} g_{\alpha\beta} \frac{\delta\xi^\alpha}{ds} \frac{dx^\beta}{ds} \delta u = 0, \Big]$$

where

$$\kappa^2 = g_{\mu\nu} \frac{\delta^2 x^\mu}{ds^2} \frac{\delta^2 x^\nu}{ds^2}, \tag{15.5}$$

$$\kappa\, \Delta\kappa = \left[g_{\alpha\beta} \left(\frac{\delta^2 \xi^\alpha}{ds^2} + R^{\alpha}_{\cdot\mu\nu\omega} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \xi^\omega \right) \frac{\delta^2 x^\beta}{ds^2} - 2\kappa^2 g_{\mu\nu} \frac{\delta\xi^\mu}{ds} \frac{dx^\nu}{ds} \right] \delta u . \tag{15.6}$$

If we assume that ξ^λ are functions of the coordinates, then we can apply either the method with the use of Δ-variation or that of D-variation and prove the following.

Theorem 17: In order that an infinitesimal deformation of a Riemannian space carry any conformal circle into conformal circle, it is necessary and sufficient that the transformation is an infinitesimal conformal transformation, that is to say, the vector ξ^λ defining the deformation satisfy the equations[22]

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2\,\phi\, g_{\mu\nu} . \tag{15.7}$$

(22) K. Yano et Tomonaga: Quelques remarques sur les groupes de transformations dans les espaces à connexion linéaire, IV. Proc. 22 (1946), 173-83.