47. A Generalization of Haar's Measure.

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The theory of Haar's measure asserts that we may introduce an invariant measure in any locally bicompact topological group G; moreover, if G is the sum of at most countable bicompact sets, the introduced measure is essentially unique and any two measurable sets having equal measures are decomposition equivalent (Zerlegungsgleich). These last two properties, however, are satisfied by the rotation invariant measure defined on the surface of the sphere of euclid spaces. The purpose of the present note is to extend Haar's theory in such a way that the above case is obtained as a special case. For the sake of shortness we only report the results; the detailed proof will be pubished elsewhere.

Le Ω be a locally bicompact uniform space defined by the system of neighbourhood $\{U_\alpha(p)\}$ and let H be a group of homeomorphisms of Ω satisfying the following two conditions:

(A) $\sigma U_\alpha(p) = U_\alpha(\sigma p)$ for $\sigma \varepsilon H$,

(B) for any two points p and q in Ω there exists an element $\sigma \varepsilon H$ sucht hat $\sigma p = q$, viz. H is transitive on Ω.

Theorem 1. There exists at least one H-invariant measure μ^* in Ω.

Let G be a topological group, then G may be considered as a group G_1 of homeomorphisms of G onto itself by corresponding to every element a of G the topological mapping $\varphi(x)=ax$. If Ω is a locally bicompact topological group G and H is G_1, we are in the case of Haar's measure. While if Ω is the surface of a sphere of euclid space and H is the rotation group, we are in the case of the rotation invariant measure.

In order to introduce a topology in the group H we define a complete system of neighbourhoods $\{V\}$ of zero of the group H. Let α be an index of neighbourhoods of Ω and let F be an arbitrary bicompact set in Ω and we define V as the totality of $\sigma \varepsilon H$ such that $\sigma p \varepsilon U_\alpha(p)$ for every point p ε F. It may easily be verified that H is then a topological group. Moreover in the case of topological group G, the topological group G_1 obtained by the above procedure is homeomeorphic with the topological group G.

Theorem 2. μ^* is a continuous measure, viz. if A is a measurable set of finite measure in Ω then, for an arbitrary positive number ε, there exists a neighbourhood V of zero of H such that $\sigma \varepsilon V$ implies $\mu(A \ominus \sigma A) < \varepsilon$, where $A \ominus \sigma A$ denotss the symmetric difference of A and σA.

Theorem 3. If $f(x)$ is a continuous function on Ω, then $f(\sigma^{-1}x)$ is also a continuous function of (x, σ) on the product space $\Omega \times H$.

Theorem 4. If Ω is the sum of a sequence of bicompact sets and if the topological group is locally bicompact and if H is expressible as the sum of at most countable bicompact sets, then the H-invariant measure in Ω is unique and any two measurable sets having equal measure are decomposition-equivalent.

Corollary. Haar's measure is unique.

In the rest of this note, we assume that Ω is the sum of at most countable bicompact sets.

Theorem 5. In order that H is totally bounded it is necessary and sufficient that Ω is totally bounded.

Theorem 6. For any open set U in H there exists at most oountable elements $a_1, a_2, \dots$ in H such that $\bigvee_{n=1}^{\infty} a_n \; U = H$.

Theorem 7. If Ω is connected or is a metric space in which evrey bounded subset is compact, then H is locally bounded.

We furthermore assume the following condition:

(C) for any point $q \varepsilon U_\alpha(p)$ there exists an index β (which may depend upon q) such that $r \varepsilon U_\beta(p)$ and $s \varepsilon U_\beta(v)$ implies $s \varepsilon U_\alpha(r)$. This condition is satisfied if Ω is a metric space or a topological group. Under the condition (C) we have the

Theorem 8. Let H be the group of all the homeomorphisms σ of Ω satisfying the condition (A), then H is complete.

Theorem 9. Let H be the group given in Theorem 8, then H is locally bicompact in the following cases:

(i) Ω is bicompact ($\overline{H}$ is bicompact in this case),

(ii) Ω is connected,

(iii) Ω is a metric space whose bounded set is compact.

In these cases if H satisfies the condition (B), the H invariant measure is unique and any two measurable sets having equal measures are dceomposition-equivalent.

Theorem 10. Let H be a group given in Theorem 1. If H is locally bicompact and if a subgroup H' of H is transitive on Ω, then the H'-invariant measure is unique and any two measurable sets having equal measures are decomposition-equivalent with respect to H'. Thus the H'-invariant measure is H-invariant.

Corollary. Lebesgue measure is invariant by rotations.