On Haar Measure of Some Groups[1)]

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In his famous paper on the theory of invariants, A. Hurwitz[2)] introduced the notion of invariant measure of group manifold. He gave an explicit expression for Haar measure of unitary unimodular groups and orthogonal groups by means of generalized polar coordinates. Afterwards H. Weyl[3)] obtained another expression. I will obtain other expressions for unitary, unitary symplectic, and orthogonal groups, using the Cayley's parametrization. Concerning unitary groups we shall prove the following:

Theorem 1. *The infinitesimal volume element $d\Omega$ of Haar measure of unitary group of n-th order is given by the following formula*

$$d\Omega = |E_n + H^2|^{-n} dh$$

where H is the Cayley's parameter of unitary matrix

$$U = (E_n + iH)(E_n - iH)^{-1}$$

and Hermitian, so that

$$\tilde{H} = H = (h_{ik}) = (a_{ik} + ib_{ik}),\ (a_{ik} = a_{ki},\ b_{ki} = -b_{ik})$$

and dh is the product of all differentials of n parameters,

$$dh = da_{11} da_{12} \dots da_{nn} db_{21} \dots db_{n,n-1}\,.$$

Proof. Let U be an unitary matrix of n-th order, which is represented by Cayley's parameters as follows:

$$U = (E_n + iH)(E_n - iH)^{-1}$$

where H is a Hermitian matrix. We form the differential of U

$$dU = \{E + i(H + dH)\}\{E - i(H + dH)\}^{-1} - (E + iH)(E - iH)^{-1},$$

then we get, by left multiplication of $E - i(H + dH)$ and right multiplication of $E - iH$,

$$\begin{aligned}&\{E - i(H + dH)\}\, dU(E - iH) \\ &\quad = \{E + i(H + dH)\}(E - iH) - \{E - i(H + dH)\}(E + iH) \\ &\quad = 2i dH.\end{aligned}$$

If we neglect the terms of 2-nd order, we obtain:

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1) A brief sketch of this paper was read at the May meeting of the Mathematical Society of Japan, 1947,

2) A. Hurwitz, Ueber die Erzeugung der Invarianten durch Intergration, Göttinger Nachrichten, 1897. S. 71–90.

3) H. Weyl, Classical groups, 1939.

$$(E-iH)dU(E-iH)=2idH,$$
$$dU=(E-iH)^{-1}2idH(E-iH)^{-1}.$$

By left translation an infinitesimal element dU is transformed to UdU,

$$dU \rightarrow UdU.$$

By right translation

$$dU \rightarrow dU \cdot U.$$

If we define the metric of dU as follows:

$$ds^2 = \sum_{i,k} |dU_{ik}|^2$$

which is evidently expressed by

$$ds^2 = T_r(\widetilde{dU} \cdot dU),$$

where $\widetilde{dU}$ denotes the Hermitian conjugate of dU. Metric thus defined is clearly invariant under the left and right translations, because

$$T_r(\widetilde{U \cdot dU} \cdot U \cdot dU) T_r(\widetilde{dU} \cdot dU)$$
$$T_r(\widetilde{dU \cdot U} \cdot dU \cdot U) = T_r(\widetilde{U} \cdot \widetilde{dU} \cdot dU \cdot U) = T_r(\widetilde{dU} \cdot dU)$$

In other words, left and right translations by any group element mean a motion in such a Riemann space. If we calculate the volume element $d\Omega$ by means of this invariant line element, we obtain the invariant measure.

$$\begin{aligned} ds^2 &= T_r(\widetilde{dU} \cdot dU) \\ &= T_r\{(E+iH)^{-1}(2idH)(H+iH)^{-1} \cdot (E-iH)(2idH)(E-iH)^{-1}\} \\ &= 4T_r\{(E+H^2)^{-1}dH(E+H^2)^{-1}dH\} \end{aligned}$$

We shall omit the unessential constant 4 and write

$$ds^2 = T_r[\{(E+H^2)^{-1}dH\}^2].$$

Calculation under this form seems very complicated, so we simplify our procedure by the following reasoning.

We consider a n^2-dimensional Euclidean space $\mathfrak{H}$ formed by H and with metric:

$$d\sigma^2 = T_r(\widetilde{dH} \cdot dH).$$

When H is transformed by any unitary matrix U as follows:

$$H \rightarrow \tilde{U}HU,$$

this transformation induces a motion also in $\mathfrak{H}$, for

$$T_r(\widetilde{\tilde{U}dHU} \cdot \tilde{U}dH \cdot U) = T_r(\widetilde{dH} \cdot dH).$$

By a well known theorem any Hermitian matrix can be transformed to a diagonal matrix L by a suitable unitary matrix W:

$$\tilde{W}HW = L = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Let dh be an infinitesimal volume element at the point H, and dh' be the image of dh at L. Then

$$dh = dh'.$$

Let $d\Omega$ resp. $d\Omega'$ be the volume element at $U=(E+iH)(E-iH)^{-1}$ resp. $(E+iL)(E-iL)^{-1}$, then we obtain similarly

$$d\Omega = d\Omega'.$$

ds'^2 at U' is evidently

$$ds'^2 = T_r((E+L^2)^{-1}dH)^2 = \sum_{i=1}^{n}\sum_{k=1}^{n}(1+\lambda_i^2)^{-1}(1+\lambda_k^2)^{-1}dh_{ik}dh_{ki}$$

$$= \Sigma\Sigma(1+\lambda_i^2)^{-1}(1+\lambda_k^2)^{-1}\,|\,dh_{ik}\,|^2$$

$$|\,dh_{ik}\,|^2 = da_{ik}^2 + db_{ik}^2$$

$$ds'^2 = \sum_{i=1}^{n}(1+\lambda_i^2)^{-2}da_{ii}^2 + 2\sum_{i<k}(1+\lambda_i^2)^{-1}(1+\lambda_k^2)^{-1}da_{ik}^2$$

$$+ 2\sum_{i<k}(1+\lambda_i^2)^{-1}(1+\lambda_k^2)^{-1}db_{ik}^2.$$

Therefore n^2 parametric curves at the point L are orthogonal to each other. Then the calculation of $\sqrt{g}$ is very easy.

$$g = \prod_{i=1}^{n}(1+\lambda_i^2)^{-2}\prod_{i<k}2(1+\lambda_i^2)^{-1}(1+\lambda_k^2)^{-1}\prod_{i<k}2(1+\lambda_i^2)^{-1}(1+\lambda_k^2)^{-1}$$

$$= 2^{n(n-1)}\prod_{i=1}^{n}(1+\lambda_i^2)^{-2n} = 2^{n(n-1)}\,|\,E+L^2\,|^{-2n}$$

$$\sqrt{g} = 2^{\frac{n(n-1)}{2}}\,|\,E+L^2\,|^{-n}.$$

Again we omit the unessential constant $2^{\frac{n(n-1)}{2}}$, then we get, remembering

$$|\,E+L^2\,| = |\,E+H^2\,|$$

$$d\Omega = d\Omega' = |\,E+L^2\,|^{-n}dh' = |\,E+H^2\,|^{-n}dh. \qquad \text{q.e.d.}$$

Now we consider the unitary symplectic group. A unitary symplectic matrix is also represented by Cayley's parameters as follows:

$$U=(E+iH)(E-iH)^{-1}.$$

where H is Hermitian and following special form[4]

$$H = \begin{pmatrix} A & B \\ \tilde{B} & -A \end{pmatrix}$$

A is a Hermitian matrix of n-th order and B is complex symmetric of n-th order. Method of calculation is the same as that of unitary matrix. Reduction to diagonal form is here the essential key point to the solution, so we employ the similar theorem due to Weyl.[5]

The same process leads to the following theorem:

4) Weyl, loc. cit. 169.
5) Weyl, loc. cit. 217.

Theorem 2. *For unitary symplectic group of 2n-th order, the Haar measure possesses the following form:*

$$d\Omega = |E+H^2|^{-(n+\frac{1}{2})}\, dh.$$

With respect to the orthogonal groups the theorem of diagonal form does not hold but almost similarly the form

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta & & & & 0 \\ \sin\theta_1 & \cos\theta_1 & & & & \\ & & \cos\theta_2 & -\sin\theta_2 & & \\ & & \sin\theta_2 & \cos\theta_2 & & \\ 0 & & & & \ddots & \end{pmatrix}$$

plays the same role. Although H itself is not diagonal and has the form

$$H = i\begin{pmatrix} 0 & \delta_1 & & & \\ -\delta_1 & 0 & & & \\ & & 0 & \delta_2 & \\ & & -\delta_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

but H^2 is diagonal

$$\begin{pmatrix} \delta_1^2 & & & & \\ & \delta_1^2 & & & \\ & & \delta_2^2 & & \\ & & & \delta_2^2 & \\ & & & & \ddots \end{pmatrix}.$$

Hence we can apply the same method.

$$ds^2 = \sum_{i \neq k}(1+\lambda_i^2)^{-1}(1+\lambda_k^2)^{-1}dh_{ik}^2 .$$

Remembering $|E+H^2| = \Pi(1+\delta_i^2)^2$

$$\sqrt{g} = |E+H^2|^{-\frac{n-1}{2}}$$

We have the following theorem:

Theorem 3. *The Haar measure of orthogonal goup of m-th order is expressed as follows:*

$$d\Omega = E+H^2|^{-\frac{n-1}{2}} dh.$$

In all these cases the exponent e in the expression $|E+H^2|^{-e}$ is the quotient

$$e = r/n ,$$

where r is the dimension of the group and n the order of matrices.